Multigrid Methods for Saddle Point Problems

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Advances in Mathematics of Finite Elements

(In honor of the ninetieth birthday of Ivo Babuška)

Outline

- Problems
- Methods
- Analysis
- Numerics
- Conclusions

References

B., Hengguang Li and Li-yeng Sung

Multigrid methods for saddle point problems: Stokes and Lamé systems

Numer. Math. (2014)

B., Duk-Soon Oh and Li-yeng Sung

Multigrid methods for saddle point problems: Darcy systems

Preprint

References

Numer. Math. 20, 179—192 (1973)

The Finite Element Method with Lagrangian Multipliers*

Ivo Babuška

Institut for Fluid Dynamic and Applied Mathematics,
University of Maryland, College Park/U.S.A.

Received January 26, 1972

Summary. The Dirichlet problem for second order differential equations is chosen as a model problem to show how the finite element method may be implemented to avoid difficulty in fulfilling essential (stable) boundary conditions. The implementation is based on the application of Lagrangian multiplier. The rate of convergence is proved.

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Analysis of Mixed Methods Using Mesh Dependent Norms*

By I. Babuška, J. Osborn and J. Pitkäranta

Abstract. This paper analyzes mixed methods for the biharmonic problem by means of new families of mesh dependent norms which are introduced and studied. More specifically, several mixed methods are shown to be stable with respect to these norms and, as a consequence, error estimates are obtained in a simple and direct manner.

1. Introduction. In [5] Brezzi studied Ritz-Galerkin approximation of saddle-point problems arising in connection with Lagrange multipliers. These problems have the form:

Given $f \in \mathcal{V}'$ and $g \in \mathcal{W}'$, find $(u, \psi) \in \mathcal{V} \times \mathcal{W}$ satisfying

$$(1.1)\qquad \begin{cases} a(u, v) + b(v, \psi) = (f, v) & \forall v \in \mathcal{V}, \\ b(u, \varphi) = (g, \varphi) & \forall \varphi \in \mathcal{W}, \end{cases}$$

Saddle Point Problems

Saddle Point Problems

Find $(u,p) \in V \times Q$ such that

$$
\begin{aligned}
a(u,v) + b(v,p) &= F(v) \qquad \forall\, v \in V \\
b(u,q) - c(p,q) &= G(q) \qquad \forall\, q \in Q
\end{aligned}
$$

V and Q are Hilbert spaces.

$a(\cdot,\cdot)$ is a bounded bilinear form on $V \times V$.

$b(\cdot,\cdot)$ is a bounded bilinear form on $V \times Q$.

$c(\cdot,\cdot)$ is a bounded bilinear form on $Q \times Q$.

F is a bounded linear function on V.

G is a bounded linear functional on Q.

Goal Construct multigrid methods that converge uniformly in the energy norm $\|\cdot\|_V + \|\cdot\|_Q$ for elliptic boundary value problems formulated as saddle point problems, without assuming full elliptic regularity.

Saddle Point Problems

Find $(u, p) \in V \times Q$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= F(v) \qquad \forall\, v \in V \\ b(u, q) - c(p, q) &= G(q) \qquad \forall\, q \in Q \end{aligned}$$

Most references in the literature (Verfürth, Wittum, Braess-Sarazin, Schöberl-Zulehner, ...) only yield convergence in norms other than the energy norm and many require full elliptic regularity (convex domain).

Saddle Point Problems

Find $(u, p) \in V \times Q$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= F(v) \qquad \forall\, v \in V \\ b(u,q) - c(p,q) &= G(q) \qquad \forall\, q \in Q \end{aligned}$$

Most references in the literature (Verfürth, Wittum, Braess-Sarazin, Schöberl-Zulehner, ...) only yield convergence in norms other than the energy norm and many require full elliptic regularity (convex domain).

We will focus on two types of saddle point problems associated with second order elliptic boundary value problems.

Saddle Point Problem I

Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= F(\boldsymbol{v}) && \forall\, \boldsymbol{v} \in V \\ b(\boldsymbol{u}, q) &= 0 && \forall\, q \in Q \end{aligned}$$

Ω is a bounded polyhedral domain in $\mathbb{R}^d$. $(d = 2, 3)$

V is a (closed) subspace of $[H^1(\Omega)]^d$.

Q is a (closed) subspace of $L_2(\Omega)$.

$a(\cdot, \cdot)$ is a symmetric bounded bilinear form on $V \times V$.

$b(\cdot, \cdot)$ is a bounded bilinear form on $V \times Q$.

F is a bounded linear functional on V.

Saddle Point Problem I

Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= F(\boldsymbol{v}) \qquad && \forall\, \boldsymbol{v} \in V \\ b(\boldsymbol{u}, q) &= 0 && \forall\, q \in Q \end{aligned}$$

There exist positive constants γ and β such that

Coercivity

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq \gamma \|\boldsymbol{v}\|_{H^1(\Omega)}^2 \qquad \forall\, \boldsymbol{v} \in V$$

Inf-Sup Condition

$$\inf_{q \in Q} \sup_{\boldsymbol{v} \in V} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H^1(\Omega)} \|q\|_{L_2(\Omega)}} \geq \beta$$

Saddle Point Problem I

Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= F(\boldsymbol{v}) && \forall\, \boldsymbol{v} \in V \\ b(\boldsymbol{u}, q) &= 0 && \forall\, q \in Q \end{aligned}$$

Under these assumptions the saddle point problem is well-posed.

Babuška 1973

Brezzi 1974

Saddle Point Problem I

Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= F(\boldsymbol{v}) \qquad && \forall\, \boldsymbol{v} \in V \\ b(\boldsymbol{u}, q) &= 0 && \forall\, q \in Q \end{aligned}$$

Elliptic Regularity

There exists $\alpha \in (0, 1]$ such that

$$\|\boldsymbol{u}\|_{H^{1+\alpha}(\Omega)} + \|p\|_{H^{\alpha}(\Omega)} \leq C_\Omega \|F\|_{H^{-1+\alpha}(\Omega)}$$

Saddle Point Problem I

Stokes System ($\boldsymbol{u} =$ fluid velocity, $p =$ pressure)

Find $(\boldsymbol{u}, p) \in [H_0^1(\Omega)]^d \times L_2^0(\Omega)$ such that

$$\begin{aligned} \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, dx - \int_\Omega (\nabla \cdot \boldsymbol{u}) p\, dx &= F(\boldsymbol{v}) && \forall\, \boldsymbol{v} \in [H_0^1(\Omega)]^d \\ - \int_\Omega (\nabla \cdot \boldsymbol{u}) q\, dx &= 0 && \forall\, q \in L_2^0(\Omega) \end{aligned}$$

Ω is a bounded polyhedral domain in $\mathbb{R}^d$. ($d = 2, 3$)

$V = [H_0^1(\Omega)]^d$ (no-slip)

$Q = L_2^0(\Omega) = \{q \in L_2(\Omega) : \int_\Omega v\, dx = 0\}$

Saddle Point Problem I

Stokes System ($\boldsymbol{u} =$ fluid velocity, $p =$ pressure)

Find $(\boldsymbol{u}, p) \in [H_0^1(\Omega)]^d \times L_2^0(\Omega)$ such that

$$\begin{aligned} \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, dx - \int_\Omega (\nabla \cdot \boldsymbol{u}) p \, dx &= F(\boldsymbol{v}) \qquad \forall \, \boldsymbol{v} \in [H_0^1(\Omega)]^d \\ - \int_\Omega (\nabla \cdot \boldsymbol{u}) q \, dx &= 0 \qquad \forall \, q \in L_2^0(\Omega) \end{aligned}$$

$$\begin{aligned} a(\boldsymbol{w}, \boldsymbol{v}) &= \int_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{v} \, dx \\ b(\boldsymbol{v}, q) &= - \int_\Omega (\nabla \cdot \boldsymbol{v}) \cdot q \, dx \end{aligned}$$

Saddle Point Problem I

Stokes System $\quad$ ($\boldsymbol{u} =$ fluid velocity, $p =$ pressure)

Find $(\boldsymbol{u}, p) \in [H_0^1(\Omega)]^d \times L_2^0(\Omega)$ such that

$$\begin{aligned} \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, dx - \int_\Omega (\nabla \cdot \boldsymbol{u}) p \, dx &= F(\boldsymbol{v}) \qquad && \forall\, \boldsymbol{v} \in [H_0^1(\Omega)]^d \\ -\int_\Omega (\nabla \cdot \boldsymbol{u}) q \, dx &= 0 && \forall\, q \in L_2^0(\Omega) \end{aligned}$$

Elliptic Regularity

There exists $\alpha \in (\frac{1}{2}, 1]$ such that

$$\|\boldsymbol{u}\|_{H^{1+\alpha}(\Omega)} + \|p\|_{H^\alpha(\Omega)} \le C_\Omega \|F\|_{H^{-1+\alpha}(\Omega)}$$

Saddle Point Problem I

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) &= F(\boldsymbol{v}) \qquad && \forall\, \boldsymbol{v} \in V_h \\ b(\boldsymbol{u}_h, q) \qquad\qquad\qquad &= 0 && \forall\, q \in Q_h \end{aligned}$$

$V_h \times Q_h \subset [H_0^1(\Omega)]^d \times L_2^0(\Omega)$ is a stable pair for the Stokes system (say the P_ℓ-$P_{\ell-1}$ ($\ell \geq 2$) Taylor-Hood elements) i.e.,

$$\inf_{q \in Q_h} \sup_{\boldsymbol{v} \in V_h} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}} \geq \beta_d > 0$$

where the discrete inf-sup constant β_d is independent of the mesh size h.

Saddle Point Problem I

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) &= F(\boldsymbol{v}) && \forall\, \boldsymbol{v} \in V_h \\ b(\boldsymbol{u}_h, q) &= 0 && \forall\, q \in Q_h \end{aligned}$$

$V_h \times Q_h \subset [H_0^1(\Omega)]^d \times L_2^0(\Omega)$ is a stable pair for the Stokes system (say the P_ℓ-$P_{\ell-1}$ ($\ell \geq 2$) Taylor-Hood elements) i.e.,

$$\inf_{q \in Q_h} \sup_{\boldsymbol{v} \in V_h} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}} \geq \beta_d > 0$$

The coercivity of $a(\cdot, \cdot)$ and the discrete inf-sup condition imply that the discrete problem provides a stable approximation of the Stokes system.

Saddle Point Problem I

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) &= F(\boldsymbol{v}) && \forall\, \boldsymbol{v} \in V_h \\ b(\boldsymbol{u}_h, q) &= 0 && \forall\, q \in Q_h \end{aligned}$$

Compact Form

$$\mathcal{B}\big((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)\big) = F(\boldsymbol{v}) \qquad \forall\, (\boldsymbol{v}, q) \in V_h \times Q_h$$

where

$$\mathcal{B}\big((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)\big) = a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) + b(\boldsymbol{u}_h, q)$$

Saddle Point Problem I

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) &= F(\boldsymbol{v}) && \forall\, \boldsymbol{v} \in V_h \\ b(\boldsymbol{u}_h, q) &= 0 && \forall\, q \in Q_h \end{aligned}$$

Compact Form

$$\mathcal{B}\big((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)\big) = F(\boldsymbol{v}) \qquad \forall\, (\boldsymbol{v}, q) \in V_h \times Q_h$$

Stability Estimate

$$\sup_{(\boldsymbol{w},r)\in V_h\times Q_h} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{\|\boldsymbol{w}\|_{H^1(\Omega)} + \|r\|_{L_2(\Omega)}} \approx \|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}$$

for all $(\boldsymbol{v}, q) \in V_h \times Q_h$

Saddle Point Problem I

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) &= F(\boldsymbol{v}) \qquad && \forall\, \boldsymbol{v} \in V_h \\ b(\boldsymbol{u}_h, q) &= 0 && \forall\, q \in Q_h \end{aligned}$$

Compact Form

$$\mathcal{B}\big((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)\big) = F(\boldsymbol{v}) \qquad \forall\, (\boldsymbol{v}, q) \in V_h \times Q_h$$

Quasi-Optimal Error Estimate

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega)} + \|p - p_h\|_{L_2(\Omega)} \\ &\qquad \le C\Big(\inf_{\boldsymbol{v} \in V_h} \|\boldsymbol{u} - \boldsymbol{v}\|_{H^1(\Omega)} + \inf_{q \in Q_h} \|p - q\|_{L_2(\Omega)} \Big) \end{aligned}$$

Saddle Point Problem I

We can also allow $a(\cdot,\cdot)$ to be nonsymmetric in the saddle point problem

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= F(\boldsymbol{v}) \qquad && \forall\, \boldsymbol{v} \in [H_0^1(\Omega)]^d \\ b(\boldsymbol{u},q) &= 0 && \forall\, q \in L_2^0(\Omega) \end{aligned}$$

Oseen System

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\,dx + \int_\Omega (\boldsymbol{w}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{v}\,dx$$

$$b(\boldsymbol{v},p) = -\int_\Omega (\nabla\cdot\boldsymbol{v})p\,dx$$

$\boldsymbol{w} \in [W_\infty^1(\Omega)]^d \cap H(\mathrm{div}^0;\Omega)$ is a wind function.

Saddle Point Problem I

We can also consider a saddle point problem of a more general form

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= F(\boldsymbol{v}) \qquad && \forall\, \boldsymbol{v} \in [H_0^1(\Omega)]^d \\ b(\boldsymbol{u}, q) - c(p, q) &= 0 && \forall\, q \in L_2^0(\Omega) \end{aligned}$$

Lamé System

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2\mu \int_\Omega \epsilon(\boldsymbol{u}) : \epsilon(\boldsymbol{v})\, dx$$

$$b(\boldsymbol{v}, p) = -\int_\Omega (\nabla \cdot v) p\, dx$$

$$c(p, q) = \frac{1}{\lambda} \int_\Omega pq\, dx$$

Saddle Point Problem II

Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= 0 && \forall\, \boldsymbol{v} \in V \\ b(\boldsymbol{u},q) &= G(q) && \forall\, q \in Q \end{aligned}$$

Ω is a bounded polyhedral domain in $\mathbb{R}^d$. $(d = 2, 3)$

V is a (closed) subspace of $[L_2(\Omega)]^d$.

Q is a (closed) subspace of $H^1(\Omega)$.

$a(\cdot,\cdot)$ is a symmetric bounded bilinear form on $V \times V$.

$b(\cdot,\cdot)$ is a bounded bilinear form on $V \times Q$.

G is a bounded linear functional on Q.

Saddle Point Problem II

Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= 0 && \forall\, \boldsymbol{v} \in V \\ b(\boldsymbol{u}, q) &= G(q) && \forall\, q \in Q \end{aligned}$$

There exist positive constants γ and β such that

Coercivity

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq \gamma \|\boldsymbol{v}\|^2_{L_2(\Omega)} \qquad \forall\, \boldsymbol{v} \in V$$

Inf-Sup Condition

$$\inf_{q \in Q} \sup_{\boldsymbol{v} \in V} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{L_2(\Omega)} \|q\|_{H^1(\Omega)}} \geq \beta$$

Saddle Point Problem II

Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= 0 && \forall\, \boldsymbol{v} \in V \\ b(\boldsymbol{u}, q) &= G(q) && \forall\, q \in Q \end{aligned}$$

Under these assumptions the saddle point problem is well-posed.

Saddle Point Problem II

Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= 0 & \forall\, \boldsymbol{v} \in V \\ b(\boldsymbol{u}, q) &= G(q) & \forall\, q \in Q \end{aligned}$$

Under these assumptions the saddle point problem is well-posed.

Elliptic Regularity

There exists $\alpha \in (0, 1]$ such that

$$\|\boldsymbol{u}\|_{H^\alpha(\Omega)} + \|p\|_{H^{1+\alpha}(\Omega)} \leq C_\Omega \|G\|_{H^{-1+\alpha}(\Omega)}$$

Saddle Point Problem II

Darcy System ($\boldsymbol{u}$ = fluid velocity, p = pressure)

Find $(\boldsymbol{u}, p) \in [L_2(\Omega)]^d \times H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,dx - \int_\Omega \boldsymbol{v}\cdot\nabla p\,dx &= 0 && \forall\,\boldsymbol{v} \in [L_2(\Omega)]^d \\ -\int_\Omega \boldsymbol{u}\cdot\nabla q\,dx &= G(q) && \forall\,q \in H_0^1(\Omega) \end{aligned}$$

Ω is a bounded polyhedral domain in $\mathbb{R}^d$. ($d = 2, 3$)

$V = [L_2(\Omega)]^d$

$Q = H_0^1(\Omega)$

K is a $d \times d$ SPD matrix.

Saddle Point Problem II

Darcy System ($\boldsymbol{u} =$ fluid velocity, $p =$ pressure)

Find $(\boldsymbol{u}, p) \in [L_2(\Omega)]^d \times H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,dx - \int_\Omega \boldsymbol{v}\cdot\nabla p\,dx &= 0 \qquad &\forall\,\boldsymbol{v} \in [L_2(\Omega)]^d \\ -\int_\Omega \boldsymbol{u}\cdot\nabla q\,dx &= G(q) \qquad &\forall\, q \in H_0^1(\Omega) \end{aligned}$$

$$\begin{aligned} a(\boldsymbol{w},\boldsymbol{v}) &= \int_\Omega K^{-1}\boldsymbol{w}\cdot\boldsymbol{v}\,dx \\ b(\boldsymbol{v},q) &= -\int_\Omega \boldsymbol{v}\cdot\nabla q\,dx \end{aligned}$$

Saddle Point Problem II

Darcy System

($\boldsymbol{u}$ = fluid velocity, p = pressure)

Find $(\boldsymbol{u}, p) \in [L_2(\Omega)]^d \times H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,dx - \int_\Omega \boldsymbol{v}\cdot\nabla p\,dx &= 0 \qquad && \forall\,\boldsymbol{v}\in[L_2(\Omega)]^d \\ -\int_\Omega \boldsymbol{u}\cdot\nabla q\,dx &= G(q) && \forall\,q\in H_0^1(\Omega) \end{aligned}$$

Elliptic Regularity

There exists $\alpha \in (\frac{1}{2}, 1]$ such that

$$\|\boldsymbol{u}\|_{H^\alpha(\Omega)} + \|p\|_{H^{1+\alpha}(\Omega)} \le C_\Omega \|G\|_{H^{-1+\alpha}(\Omega)}$$

Saddle Point Problem II

Darcy System ($\boldsymbol{u} =$ fluid velocity, $p =$ pressure)

Find $(\boldsymbol{u}, p) \in [L_2(\Omega)]^d \times H_0^1(\Omega)$ such that

$$
\begin{aligned}
\int_\Omega K^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,dx - \int_\Omega \boldsymbol{v}\cdot\nabla p\,dx &= 0 \qquad && \forall\,\boldsymbol{v} \in [L_2(\Omega)]^d \\
-\int_\Omega \boldsymbol{u}\cdot\nabla q\,dx &= G(q) \qquad && \forall\, q \in H_0^1(\Omega)
\end{aligned}
$$

The dual problem posed on $H(\mathrm{div};\Omega) \times L_2(\Omega)$ is the more attractive formulation of the Darcy system since it provides more information on the velocity $\boldsymbol{u}$.

Saddle Point Problem II

Darcy System

($\boldsymbol{u}$ = fluid velocity, p = pressure)

Find $(\boldsymbol{u}, p) \in [L_2(\Omega)]^d \times H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,dx - \int_\Omega \boldsymbol{v}\cdot\nabla p\,dx &= 0 \qquad && \forall\,\boldsymbol{v} \in [L_2(\Omega)]^d \\ -\int_\Omega \boldsymbol{u}\cdot\nabla q\,dx &= G(q) \qquad && \forall\,q \in H_0^1(\Omega) \end{aligned}$$

Dual Formulation

Find $(\boldsymbol{u}, p) \in H(\mathrm{div};\Omega) \times L_2(\Omega)$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,dx + \int_\Omega (\nabla\cdot\boldsymbol{v})p\,dx &= 0 \qquad && \forall\,\boldsymbol{v} \in H(\mathrm{div};\Omega) \\ \int_\Omega (\nabla\cdot\boldsymbol{u})q\,dx &= G(q) \qquad && \forall\,q \in L_2(\Omega) \end{aligned}$$

Saddle Point Problem II

Darcy System ($\boldsymbol{u}$ = fluid velocity, p = pressure)

Find $(\boldsymbol{u}, p) \in [L_2(\Omega)]^d \times H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,dx - \int_\Omega \boldsymbol{v}\cdot\nabla p\,dx &= 0 && \forall\,\boldsymbol{v}\in[L_2(\Omega)]^d \\ -\int_\Omega \boldsymbol{u}\cdot\nabla q\,dx &= G(q) && \forall\,q\in H_0^1(\Omega) \end{aligned}$$

However we can treat conforming mixed finite element methods for the dual formulation posed on $H(\mathrm{div};\Omega)\times L_2(\Omega)$ as nonconforming mixed finite element methods for the formulation posed on $[L_2(\Omega)]^d \times H_0^1(\Omega)$.

Saddle Point Problem II

Darcy System ($\boldsymbol{u} =$ fluid velocity, $p =$ pressure)

Find $(\boldsymbol{u}, p) \in [L_2(\Omega)]^d \times H_0^1(\Omega)$ such that

$$
\begin{aligned}
\int_\Omega K^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\,dx - \int_\Omega \boldsymbol{v}\cdot\nabla p\,dx &= 0 \qquad && \forall\,\boldsymbol{v} \in [L_2(\Omega)]^d \\
-\int_\Omega \boldsymbol{u}\cdot\nabla q\,dx &= G(q) \qquad && \forall\,q \in H_0^1(\Omega)
\end{aligned}
$$

However we can treat conforming mixed finite element methods for the dual formulation posed on $H(\mathrm{div};\Omega) \times L_2(\Omega)$ as nonconforming mixed finite element methods for the formulation posed on $[L_2(\Omega)]^d \times H_0^1(\Omega)$.

This idea was introduced in 1980 by Babuška-Osborn-Pitkäranta for the analysis of mixed finite element methods for the biharmonic equation.

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned}
\int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\, dx + \int_\Omega (\nabla \cdot \boldsymbol{v}) p_h\, dx &= 0 && \forall\, \boldsymbol{v} \in V_h \\
\int_\Omega (\nabla \cdot \boldsymbol{u}_h) q\, dx &= G(q) && \forall\, q \in Q_h
\end{aligned}$$

$V_h = RT_\ell\,(\subset H(\mathrm{div};\Omega))$ is the Raviart-Thomas finite element space of order $\ell \geq 1$.

Q_h is the space of (discontinuous) piecewise P_ℓ functions.

$V_h \times Q_h$ is a stable finite element pair for the Darcy system posed on $H(\mathrm{div};\Omega) \times L_2(\Omega)$ and the discrete problem is the standard Raviart-Thomas finite element method for this formulation of the Darcy system.

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla \cdot \boldsymbol{v})p_h\,dx &= 0 && \forall\, \boldsymbol{v} \in V_h \\ \int_\Omega (\nabla \cdot \boldsymbol{u}_h) q\,dx &= G(q) && \forall\, q \in Q_h \end{aligned}$$

$V_h = RT_\ell\,(\subset H(\mathrm{div};\Omega))$ is the Raviart-Thomas finite element space of order $\ell \geq 1$.

Q_h is the space of (discontinuous) piecewise P_ℓ functions.

We will treat the discrete problem as a nonconforming method for the Darcy system posed on $[L_2(\Omega)]^d \times H_0^1(\Omega)$ by introducing mesh-dependent norms on V_h and Q_h.

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned}
\int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\, dx + \int_\Omega (\nabla \cdot \boldsymbol{v}) p_h\, dx &= 0 && \forall\, \boldsymbol{v} \in V_h \\
\int_\Omega (\nabla \cdot \boldsymbol{u}_h) q\, dx &= G(q) && \forall\, q \in Q_h
\end{aligned}$$

Mesh-Dependent L_2 Norm for V_h

$$\|\boldsymbol{v}\|^2_{L_2(\Omega;\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} \|\boldsymbol{v}\|^2_{L_2(T)} + \sum_{\sigma\in\mathfrak{S}_h} h_\sigma \|\boldsymbol{v}\cdot\boldsymbol{n}_\sigma\|^2_{L_2(\sigma)}$$

$\mathfrak{S}_h$ is the set of the sides (edges/faces) of the elements in $\mathcal{T}_h$.

$\boldsymbol{n}_\sigma$ is a unit normal of σ and h_σ is the diameter of σ.

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla\cdot\boldsymbol{v})p_h\,dx &= 0 & \forall\,\boldsymbol{v} \in V_h \\ \int_\Omega (\nabla\cdot\boldsymbol{u}_h)q\,dx &= G(q) & \forall\,q \in Q_h \end{aligned}$$

Mesh-Dependent L_2 Norm for V_h

$$\|\boldsymbol{v}\|^2_{L_2(\Omega;\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} \|\boldsymbol{v}\|^2_{L_2(T)} + \sum_{\sigma\in\mathfrak{S}_h} h_\sigma \|\boldsymbol{v}\cdot\boldsymbol{n}_\sigma\|^2_{L_2(\sigma)}$$

This norm is well-defined for the velocity $\boldsymbol{u} = K\nabla p \in H^\alpha(\Omega)$ ($\alpha > \frac{1}{2}$) in the continuous problem and it is also equivalent to $\|\cdot\|_{L_2(\Omega)}$ on V_h.

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla\cdot\boldsymbol{v})p_h\,dx &= 0 & \forall\,\boldsymbol{v}\in V_h \\ \int_\Omega (\nabla\cdot\boldsymbol{u}_h)q\,dx &= G(q) & \forall\,q\in Q_h \end{aligned}$$

Mesh-Dependent H^1 Norm for Q_h

$$\|q\|^2_{H^1(\Omega;\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} \|\nabla q\|^2_{L_2(T)} + \sum_{\sigma\in\mathfrak{S}_h} \frac{1}{h_\sigma}\|[\![q]\!]_\sigma\|^2_{L_2(\sigma)}$$

$[\![q]\!]_\sigma$ (a vector) is the jump of q across the side σ.

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla\cdot\boldsymbol{v})p_h\,dx &= 0 && \forall\,\boldsymbol{v} \in V_h \\ \int_\Omega (\nabla\cdot\boldsymbol{u}_h)q\,dx &= G(q) && \forall\,q \in Q_h \end{aligned}$$

Mesh-Dependent H^1 Norm for Q_h

$$\|q\|^2_{H^1(\Omega;\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} \|\nabla q\|^2_{L_2(T)} + \sum_{\sigma\in\mathfrak{S}_h} \frac{1}{h_\sigma}\|[\![q]\!]_\sigma\|^2_{L_2(\sigma)}$$

This is a standard DG norm for piecewise H^1 functions, which is well-defined on the pressure p in the continuous problem.

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla\cdot\boldsymbol{v})p_h\,dx &= 0 && \forall\,\boldsymbol{v}\in V_h \\ \int_\Omega (\nabla\cdot\boldsymbol{u}_h)q\,dx &= G(q) && \forall\,q\in Q_h \end{aligned}$$

The bilinear form

$$(\boldsymbol{w},\boldsymbol{v}) \mapsto \int_\Omega K^{-1}\boldsymbol{w}\cdot\boldsymbol{v}\,dx$$

is clearly bounded with respect to

$$\|\boldsymbol{v}\|_{L_2(\Omega;\mathcal{T}_h)} = \Big(\sum_{T\in\mathcal{T}_h} \|\boldsymbol{v}\|^2_{L_2(T)} + \sum_{\sigma\in\mathfrak{S}_h} h_\sigma \|\boldsymbol{v}\cdot\boldsymbol{n}_\sigma\|^2_{L_2(\sigma)}\Big)^{\frac{1}{2}}$$

on $\langle\boldsymbol{u}\rangle + V_h$.

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla\cdot\boldsymbol{v})p_h\,dx &= 0 && \forall\,\boldsymbol{v}\in V_h \\ \int_\Omega (\nabla\cdot\boldsymbol{u}_h)q\,dx &= G(q) && \forall\,q\in Q_h \end{aligned}$$

The bilinear form

$$(\boldsymbol{w},\boldsymbol{v}) \mapsto \int_\Omega K^{-1}\boldsymbol{w}\cdot\boldsymbol{v}\,dx$$

is also coercive on V_h:

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{v}\cdot\boldsymbol{v}\,dx &\geq \lambda_{\min}(K^{-1})\|\boldsymbol{v}\|^2_{L_2(\Omega)} \\ &\approx \|\boldsymbol{v}\|^2_{L_2(\Omega;\mathcal{T}_h)} && \forall\,\boldsymbol{v}\in V_h \end{aligned}$$

Saddle Point Problem II

It follows from the integration by parts formula

$$\int_\Omega (\nabla \cdot \boldsymbol{v}) q \, dx = \sum_{T \in \mathcal{T}_h} \Big(\int_{\partial T} (\boldsymbol{n} \cdot \boldsymbol{v}) q \, ds - \int_T \boldsymbol{v} \cdot \nabla q \, dx \Big)$$

$$= \sum_{\sigma \in \mathfrak{S}_h} \int_\sigma \boldsymbol{v} \cdot [\![q]\!]_\sigma \, ds - \sum_{T \in \mathcal{T}_h} \int_T \boldsymbol{v} \cdot \nabla q \, dx$$

that the bilinear form

$$(\boldsymbol{v}, q) \mapsto \int_\Omega (\nabla \cdot \boldsymbol{v}) q \, dx$$

is bounded on $(\langle \boldsymbol{u} \rangle + V_h) \times (\langle p \rangle + Q_h)$ with respect to

$$\|\boldsymbol{v}\|_{L_2(\Omega;\mathcal{T}_h)} = \Big(\sum_{T \in \mathcal{T}_h} \|\boldsymbol{v}\|^2_{L_2(T)} + \sum_{\sigma \in \mathfrak{S}_h} h_\sigma \|\boldsymbol{v} \cdot \boldsymbol{n}_\sigma\|^2_{L_2(\sigma)} \Big)^{\frac{1}{2}}$$

$$\|q\|_{H^1(\Omega;\mathcal{T}_h)} = \Big(\sum_{T \in \mathcal{T}_h} \|\nabla q\|^2_{L_2(T)} + \sum_{\sigma \in \mathfrak{S}_h} \frac{1}{h_\sigma} \|[\![q]\!]_\sigma\|^2_{L_2(\sigma)} \Big)^{\frac{1}{2}}$$

Saddle Point Problem II

It follows from the integration by parts formula

$$\begin{aligned}\int_\Omega (\nabla \cdot \boldsymbol{v}) q\, dx &= \sum_{T\in\mathcal{T}_h} \Big(\int_{\partial T} (\boldsymbol{n}\cdot\boldsymbol{v}) q\, ds - \int_T \boldsymbol{v}\cdot\nabla q\, dx \Big) \\ &= \sum_{\sigma\in\mathfrak{S}_h} \int_\sigma \boldsymbol{v}\cdot [\![q]\!]_\sigma\, ds - \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{v}\cdot\nabla q\, dx\end{aligned}$$

that the bilinear form

$$(\boldsymbol{v}, q) \mapsto \int_\Omega (\nabla\cdot\boldsymbol{v}) q\, dx$$

satisfies an inf-sup condition

$$\inf_{q\in Q_h} \sup_{\boldsymbol{v}\in V_h} \frac{\displaystyle\int_\Omega (\nabla\cdot\boldsymbol{v}) q\, dx}{\|\boldsymbol{v}\|_{L_2(\Omega;\mathcal{T}_h)} \|q\|_{H^1(\Omega;\mathcal{T}_h)}} \geq \beta_d > 0$$

where β_d is independent of h.

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} \int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla \cdot \boldsymbol{v}) p_h\,dx &= 0 \qquad && \forall\, \boldsymbol{v} \in V_h \\ \int_\Omega (\nabla \cdot \boldsymbol{u}_h) q\,dx &= G(q) && \forall\, q \in Q_h \end{aligned}$$

Compact Form

$$\mathcal{B}\big((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)\big) = G(q) \qquad \forall\, q \in Q_h$$

where

$$\begin{aligned} \mathcal{B}\big((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)\big) = \int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla \cdot \boldsymbol{v}) p_h\,dx \\ + \int_\Omega (\nabla \cdot \boldsymbol{u}_h) q\,dx \end{aligned}$$

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned}
\int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla\cdot\boldsymbol{v})p_h\,dx &= 0 \qquad && \forall\,\boldsymbol{v} \in V_h \\
\int_\Omega (\nabla\cdot\boldsymbol{u}_h)q\,dx &= G(q) \qquad && \forall\,q \in Q_h
\end{aligned}$$

Stability Estimate

$$\sup_{(\boldsymbol{w},r)\in V_h\times Q_h} \frac{\mathcal{B}\big((\boldsymbol{v},q),(\boldsymbol{w},r)\big)}{\|\boldsymbol{w}\|_{L_2(\Omega;\mathcal{T}_h)} + \|r\|_{H^1(\Omega;\mathcal{T}_h)}} \approx \|\boldsymbol{v}\|_{L_2(\Omega;\mathcal{T}_h)} + \|q\|_{H^1(\Omega;\mathcal{T}_h)}$$

for all $(\boldsymbol{v}, q) \in V_h \times Q_h$

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$
\begin{aligned}
\int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla\cdot\boldsymbol{v})p_h\,dx &= 0 && \forall\,\boldsymbol{v}\in V_h \\
\int_\Omega (\nabla\cdot\boldsymbol{u}_h)q\,dx &= G(q) && \forall\,q\in Q_h
\end{aligned}
$$

Quasi-Optimal Error Estimate

$$
\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega;\mathcal{T}_h)} + \|p-p_h\|_{H^1(\Omega;\mathcal{T}_h)} \\
&\qquad \le C\Big(\inf_{\boldsymbol{v}\in V_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{L_2(\Omega;\mathcal{T}_h)} + \inf_{q\in Q_h}\|p-q\|_{H^1(\Omega;\mathcal{T}_h)}\Big)
\end{aligned}
$$

Saddle Point Problem II

Discrete Problem

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned}
\int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla \cdot \boldsymbol{v}) p_h\,dx &= 0 & \forall\, \boldsymbol{v} \in V_h \\
\int_\Omega (\nabla \cdot \boldsymbol{u}_h) q\,dx &= G(q) & \forall\, q \in Q_h
\end{aligned}$$

Quasi-Optimal Error Estimate

$$\begin{aligned}
&\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L_2(\Omega;\mathcal{T}_h)} + \|p - p_h\|_{H^1(\Omega;\mathcal{T}_h)} \\
&\qquad \le C\Big(\inf_{\boldsymbol{v}\in V_h} \|\boldsymbol{u} - \boldsymbol{v}\|_{L_2(\Omega;\mathcal{T}_h)} + \inf_{q\in Q_h} \|p - q\|_{H^1(\Omega;\mathcal{T}_h)}\Big)
\end{aligned}$$

This is the reason why we do not use the lowest order Raviart-Thomas finite element pair, where Q_h is the space of piecewise constant functions, because in that case $\inf_{q\in Q_h} \|p-q\|_{H^1(\Omega;\mathcal{T}_h)}$ does not go to 0 as h decreases to 0.

Saddle Point Problem II

It follows from the definitions of the mesh-dependent norms

$$\|\boldsymbol{v}\|_{L_2(\Omega;\mathcal{T}_h)} = \Big(\sum_{T\in\mathcal{T}_h} \|\boldsymbol{v}\|^2_{L_2(T)} + \sum_{\sigma\in\mathfrak{S}_h} h_\sigma \|\boldsymbol{v}\cdot\boldsymbol{n}_\sigma\|^2_{L_2(\sigma)}\Big)^{\frac{1}{2}}$$

$$\|q\|_{H^1(\Omega;\mathcal{T}_h)} = \Big(\sum_{T\in\mathcal{T}_h} \|\nabla q\|^2_{L_2(T)} + \sum_{\sigma\in\mathfrak{S}_h} \frac{1}{h_\sigma} \|[\![q]\!]_\sigma\|^2_{L_2(\sigma)}\Big)^{\frac{1}{2}}$$

and a Poincaré-Friedrichs inequality for piecewise H^1 functions that

$$\begin{aligned} &\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)} + \|p-p_h\|_{L_2(\Omega)} \\ &\qquad\qquad \le C\big(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega;\mathcal{T}_h)} + \|p-p_h\|_{H^1(\Omega;\mathcal{T}_h)}\big) \end{aligned}$$

Hence the standard error estimate for the Raviart-Thomas finite element method follows from the error estimate in the mesh-dependent norms.

Saddle Point Problem II

We can also consider a more general (nonsymmetric) discrete problem.

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\int_\Omega K^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}\, dx + \int_\Omega (\nabla \cdot \boldsymbol{v}) p_h\, dx = 0 \qquad \forall\, \boldsymbol{v} \in V_h$$

$$\int_\Omega (\nabla \cdot \boldsymbol{u}_h) q\, dx - \int_\Omega (cp + \boldsymbol{b} \cdot \nabla_h p) q\, dx = G(q) \qquad \forall\, q \in Q_h$$

where $c \in W^1_\infty(\Omega)$, $\boldsymbol{b} \in [W^{1,\infty}(\Omega)]^d \cap H(\mathrm{div}^0; \Omega)$ and

$$c - \frac{1}{2}\nabla \cdot \boldsymbol{b} \geq 0$$

(Darcy system with convective and reactive terms)

Multigrid Methods

A Symmetric Positive Definite Problem

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall \, v \in H_0^1(\Omega)$$

Ω is a polygonal domain in $\mathbb{R}^2$.

f belongs to $L_2(\Omega)$.

Elliptic Regularity

$$\|u\|_{H^{1+\alpha}(\Omega)} \leq C_\Omega \|f\|_{H^{-1+\alpha}(\Omega)}$$

for some $\alpha \in (\frac{1}{2}, 1]$. $\quad$ ($\alpha = 1$ for convex Ω)

A Symmetric Positive Definite Problem

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^1(\Omega)$$

Set-Up for Multigrid

$\mathcal{T}_0$ is a triangulation of Ω.

$\mathcal{T}_1, \mathcal{T}_2, \ldots$ are generated from $\mathcal{T}_0$ by uniform subdivision.

h_k is the mesh size.

$V_0 \subset V_1 \subset \cdots$ are nested P_1 finite element spaces.

A Symmetric Positive Definite Problem

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^1(\Omega)$$

A Mesh-Dependent Inner Product

$$[v, w]_k = h_k^2 \sum_p v(p) w(p) \qquad \forall\, v \in V_k$$

where the summation is over all the vertices of $\mathcal{T}_k$.

$$[v, v]_k \approx \|v\|_{L_2(\Omega)}^2 \qquad \forall\, v \in V_k$$

A Symmetric Positive Definite Problem

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall \, v \in H_0^1(\Omega)$$

k-th Level Problem

$$(*) \quad A_k u_k = f_k$$

where

$$[A_k w, v]_k = a(v, w) = \int_\Omega \nabla w \cdot \nabla v \, dx \qquad \forall \, v, w \in V_k$$

$$[f_k, v]_k = \int_\Omega f v \, dx \qquad \forall \, v \in V_k$$

Multigrid Algorithms

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^1(\Omega)$$

k-th Level Problem

$$(*) \quad A_k u_k = f_k$$

- Apply pre-smoothing steps with initial guess v_0 to obtain an approximate solution $\boldsymbol{v}_\dagger$ of $(*)$.
- Transfer the residual of $\boldsymbol{v}_\dagger$ to a coarse grid and apply the multigrid algorithm on the coarse grid to find an approximate correction of $\boldsymbol{v}_\dagger$.
- Apply post-smoothing steps to the corrected approximate solution of $(*)$ to obtain the final output.

Multigrid Algorithms

Two Ingredients

- Intergrid transfer operators
- Efficient smoother

Multigrid Algorithms

Two Ingredients

- Intergrid transfer operators
- Efficient smoother

Coarse-to-Fine Operator

$I_{k-1}^{k} : V_{k-1} \longrightarrow V_k$ is the natural injection.

(The finite element spaces are nested.)

Multigrid Algorithms

Two Ingredients

- Intergrid transfer operators
- Efficient smoother

Coarse-to-Fine Operator

$I_{k-1}^{k} : V_{k-1} \longrightarrow V_k$ is the natural injection.

Fine-to-Coarse Operator

$I_k^{k-1} : V_k \longrightarrow V_{k-1}$ is the transpose of I_{k-1}^{k} with respect to the mesh-dependent inner product:

$$[I_k^{k-1} v, w]_{k-1} = \big[v, I_{k-1}^{k} w\big]_k \qquad \forall\, v \in V_k,\ w \in V_{k-1}$$

Multigrid Algorithms

Two Ingredients

- Intergrid transfer operators
- Efficient smoother

Coarse-to-Fine Operator

$I_{k-1}^{k} : V_{k-1} \longrightarrow V_k$ is the natural injection.

Ritz Projection Operator

$P_k^{k-1} : V_k \longrightarrow V_{k-1}$ is the transpose of I_{k-1}^{k} with respect to the variational bilinear form $a(\cdot,\cdot)$.

$$a\big(P_k^{k-1} v, w\big) = a\big(v, I_{k-1}^{k} w\big) \qquad \forall\, v \in V_k,\, w \in V_{k-1}$$

Multigrid Algorithms

Smoothing Step for $A_k u_k = f_k$

$$v_{\text{new}} = v_{\text{old}} + S_k(f_k - A_k v_{\text{old}})$$

Multigrid Algorithms

Smoothing Step for $A_k u_k = f_k$

$$v_{\text{new}} = v_{\text{old}} + S_k(f_k - A_k v_{\text{old}})$$

Richardson Relaxation

$$S_k = \gamma_k Id_k$$

where the damping factor $\gamma_k = Ch_k^2$ is chosen so that

the spectral radius of $S_k A_k$ is ≤ 1

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

m pre-smoothing steps

m post-smoothing steps

Exact solve on the coarse grid

$$R_k = Id_k - S_k A_k = Id_k - \gamma_k A_k$$

measures the effect of one smoothing step.

$$Id_k - I_{k-1}^k P_k^{k-1}$$

measures the effect of the exact coarse grid solve.

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

Estimates for R_k^m (smoothing property) and $Id_k - I_{k-1}^k P_k^{k-1}$ (approximation property) can be established in terms of a scale of mesh-dependent norms.

Bank and Dupont 1981

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

Mesh-Dependent Norms

$$|\!|\!| v |\!|\!|_{s,k} = \left[A_k^s v, v \right]_k \qquad \forall\, v \in V_k$$

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

Mesh-Dependent Norms

$$|\!|\!| v |\!|\!|_{s,k} = \left[A_k^s v, v \right]_k \qquad \forall\, v \in V_k$$

Connection to Sobolev Norms

$$|\!|\!| v |\!|\!|_{0,k} \approx \| v \|_{L_2(\Omega)} \qquad \forall\, v \in V_k$$

$$|\!|\!| v |\!|\!|_{1,k} = | v |_{H^1(\Omega)} \qquad \forall\, v \in V_k$$

Interpolation between Hilbert Scales

$$|\!|\!| v |\!|\!|_{s,k} \approx \| v \|_{H^s(\Omega)} \qquad \forall\, v \in V_k$$

for $0 \leq s \leq 1$ $(s \neq \frac{1}{2})$

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

Mesh-Dependent Norms

$$|||v|||_{s,k} = \left[A_k^s v, v\right]_k \qquad \forall\, v \in V_k$$

Smoothing Property

$$|||R_k^m v|||_{s,k} \lesssim (h_k \sqrt{m})^{s-t} |||v|||_{t,k} \qquad \forall\, v \in V_k$$

for $0 \le s \le t \le 2$

$$R_k = Id_k - \gamma_k A_k$$

$$\gamma_k = C h_k^2 \quad \text{and} \quad \rho(\gamma_k A_k) \le 1$$

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

Mesh-Dependent Norms

$$|\!|\!| v |\!|\!|_{s,k} = \left[A_k^s v, v \right]_k \qquad \forall\, v \in V_k$$

Approximation Property

$$\begin{aligned} |\!|\!| (Id_k - I_{k-1}^k P_k^{k-1}) v |\!|\!|_{1-\alpha,k} & \\ &\approx \| (Id_k - I_{k-1}^k P_k^{k-1}) v \|_{H^{1-\alpha}(\Omega)} \\ &\lesssim h_k^{\alpha} \| v \|_{H^1(\Omega)} \\ &\lesssim h_k^{\alpha} |\!|\!| v |\!|\!|_{1,k} \qquad \forall\, v \in V_k \end{aligned}$$

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

$$|E_k v|_{H^1(\Omega)} = \|E_k v\|_{1,k}$$

definition of mesh-dependent norm

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

$$\begin{aligned} |E_k v|_{H^1(\Omega)} &= \|E_k v\|_{1,k} \\ &= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} \end{aligned}$$

$$Id_k - I_{k-1}^k P_k^{k-1} = (Id_k - I_{k-1}^k P_k^{k-1})^2$$

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

$$\begin{aligned}
|E_k v|_{H^1(\Omega)} &= \|E_k v\|_{1,k} \\
&= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} \\
&\lesssim (h_k \sqrt{m})^{-\alpha} \|(Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1-\alpha,k}
\end{aligned}$$

smoothing property

$$\|R_k^m v\|_{1,k} \lesssim (h_k \sqrt{m})^{-\alpha} \|v\|_{1-\alpha,k}$$

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

$$\begin{aligned}
|E_k v|_{H^1(\Omega)} &= \|E_k v\|_{1,k} \\
&= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} \\
&\lesssim (h_k \sqrt{m})^{-\alpha} \|(Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1-\alpha,k} \\
&\lesssim (h_k \sqrt{m})^{-\alpha} h_k^\alpha \|(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k}
\end{aligned}$$

approximation property

$$\|(Id_k - I_{k-1}^k P_k^{k-1}) v\|_{1-\alpha,k} \lesssim h_k^\alpha \|v\|_{1,k}$$

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

$$\begin{aligned}
|E_k v|_{H^1(\Omega)} &= \|\!| E_k v \|\!|_{1,k} \\
&= \|\!| R_k^m (Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v \|\!|_{1,k} \\
&\lesssim (h_k \sqrt{m})^{-\alpha} \|\!| (Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v \|\!|_{1-\alpha,k} \\
&\lesssim (h_k \sqrt{m})^{-\alpha} h_k^\alpha \|\!| (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v \|\!|_{1,k} \\
&\lesssim (h_k \sqrt{m})^{-\alpha} h_k^\alpha h_k^\alpha \|\!| R_k^m v \|\!|_{1+\alpha,k}
\end{aligned}$$

approximation property

$$\|\!| (Id_k - I_{k-1}^k P_k^{k-1}) v \|\!|_{1,k} \lesssim h_k^\alpha \|\!| v \|\!|_{1+\alpha,k}$$

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

$$\begin{aligned}
|E_k v|_{H^1(\Omega)} &= \|E_k v\|_{1,k} \\
&= \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} \\
&\lesssim (h_k\sqrt{m})^{-\alpha} \|(Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1-\alpha,k} \\
&\lesssim (h_k\sqrt{m})^{-\alpha} h_k^\alpha \|(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{1,k} \\
&\lesssim (h_k\sqrt{m})^{-\alpha} h_k^\alpha h_k^\alpha \|R_k^m v\|_{1+\alpha,k} \\
&\lesssim (h_k\sqrt{m})^{-\alpha} h_k^\alpha h_k^\alpha (h_k\sqrt{m})^{-\alpha} \|v\|_{1,k}
\end{aligned}$$

smoothing property

$$\|R_k^m v\|_{1+\alpha,k} \lesssim (h_k\sqrt{m})^{-\alpha} \|v\|_{1,k}$$

Error Propagation for the Two-Grid Algorithm

$$E_k = R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m$$

$$\begin{aligned}
|E_k v|_{H^1(\Omega)} &= \|\!|E_k v|\!\|_{1,k} \\
&= \|\!|R_k^m (Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v|\!\|_{1,k} \\
&\lesssim (h_k\sqrt{m})^{-\alpha} \|\!|(Id_k - I_{k-1}^k P_k^{k-1})(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v|\!\|_{1-\alpha,k} \\
&\lesssim (h_k\sqrt{m})^{-\alpha} h_k^\alpha \|\!|(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v|\!\|_{1,k} \\
&\lesssim (h_k\sqrt{m})^{-\alpha} h_k^\alpha h_k^\alpha \|\!|R_k^m v|\!\|_{1+\alpha,k} \\
&\lesssim (h_k\sqrt{m})^{-\alpha} h_k^\alpha h_k^\alpha (h_k\sqrt{m})^{-\alpha} \|\!|v|\!\|_{1,k} \\
&\lesssim m^{-\alpha} |v|_{H^1(\Omega)}
\end{aligned}$$

definition of mesh-dependent norm

Convergence Results

It follows from the estimate

$$|E_k v|_{H^1(\Omega)} \lesssim m^{-\alpha}|v|_{H^1(\Omega)}$$

that the two-grid algorithm is a contraction with a contraction number bounded uniformly away from 1 provided the number of smoothing steps (independent of mesh levels) is sufficiently large.

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Bank and Dupont 1981

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The same convergence result holds for the W-cycle algorithm by a perturbation argument.

This result can be improved to uniform convergence with 1 smoothing step for both W-cycle and V-cycle algorithms by exploiting a strengthened Cauchy-Schwarz inequality.

Bramble, Pasciak, Wang, Xu, Zhang, Zikatanov,...

Summary

The key to the multigrid convergence analysis for second order symmetric positive definite problems is the existence of a scale of mesh dependent norms related to the operator $S_k A_k$, where S_k appears in the smoothing step

$$v_{\text{new}} = v_{\text{old}} - S_k(f - A_k v_{\text{old}})$$

This scale of mesh dependent norms should be equivalent to the scale of Sobolev norms between $L_2(\Omega)$ and $H^1(\Omega)$.

Then we can establish the smoothing property for the operator

$$R_k = Id_k - S_k A_k$$

and the approximation property for the operator

$$Id_k - I_{k-1}^k P_k^{k-1}$$

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Note that we can prove convergence of the multigrid methods in the energy norm (H^1 norm) because it is equivalent to the $(1, k)$ mesh-dependent norm.

Multigrid Methods for Saddle Point Problems

$\mathcal{T}_0$ is a triangulation of Ω.

$\mathcal{T}_1, \mathcal{T}_2, \ldots$ are generated from $\mathcal{T}_0$ by uniform subdivision.

h_k is the mesh size.

$V_0 \times Q_0 \subset V_1 \times Q_1 \subset \cdots$ are nested finite element spaces.

(Taylor-Hood for Stokes/Raviart-Thomas for Darcy)

Multigrid Methods for Saddle Point Problems

k-th Level Discrete Problem

Find $(\boldsymbol{u}_k, p_k) \in V_k \times Q_k$ such that

$$\mathcal{B}\big((\boldsymbol{u}_k, p_k), (\boldsymbol{v}, q)\big) = F(\boldsymbol{v}) + G(q) \qquad \forall\, (\boldsymbol{v}, q) \in V_k \times Q_k$$

The bilinear form $\mathcal{B}(\cdot, \cdot)$ satisfies the stability estimate

$$\sup_{(\boldsymbol{w}, r) \in V_k \times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{\|\boldsymbol{w}\|_{V_k} + \|r\|_{Q_k}} \approx \|\boldsymbol{v}\|_{V_k} + \|q\|_{Q_k}$$

for all $(\boldsymbol{v}, q) \in V_k \times Q_k$.

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Stokes

$$\|\boldsymbol{v}\|_{V_k} = \|\boldsymbol{v}\|_{H^1(\Omega)} \quad \text{and} \quad \|q\|_{Q_k} = \|q\|_{L_2(\Omega)}$$

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Darcy

$$\|\boldsymbol{v}\|_{V_k} = \|\boldsymbol{v}\|_{L_2(\Omega)} \quad \text{and} \quad \|q\|_{Q_k} = \|q\|_{H^1(\Omega;\mathcal{T}_h)}$$

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System Operator B_k

We can represent the bilinear form $\mathcal{B}(\cdot,\cdot)$ on $V_k \times Q_k$ by the operator $B_k : V_k \times Q_k \longrightarrow V_k \times Q_k$ defined by

$$\big[B_k(\boldsymbol{v}, q), (\boldsymbol{w}, r)\big]_k = \mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big) \qquad \forall\, (\boldsymbol{v}, q), (\boldsymbol{w}, r) \in V_k \times Q_k$$

where $[\cdot,\cdot]_k$ is a mesh-dependent inner product.

Multigrid Methods for Saddle Point Problems

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We can rewrite the k-th level discrete problem as

$$B_k(\boldsymbol{u}_k, p_k) = (\boldsymbol{f}_k, g_k)$$

where $\big[(\boldsymbol{f}_k, g_k), (\boldsymbol{w}, r)\big]_k = F(\boldsymbol{w}) + G(r) \;\; \forall\, (\boldsymbol{w}, r) \in V_k \times Q_k$.

Multigrid Methods for Saddle Point Problems

The mesh-dependent inner product $[\cdot,\cdot]_k$ on $V_k \times Q_k$ satisfies the following norm equivalence.

Stokes

$$\big[(\boldsymbol{v},q),(\boldsymbol{v},q)\big]_k \approx \|\boldsymbol{v}\|^2_{L_2(\Omega)} + h_k^2\|q\|^2_{L_2(\Omega)}$$

Darcy

$$\big[(\boldsymbol{v},q),(\boldsymbol{v},q)\big]_k \approx h_k^2\|\boldsymbol{v}\|^2_{L_2(\Omega)} + \|q\|^2_{L_2(\Omega)}$$

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For the Stokes problem, the elliptic regularity result holds only for the $\boldsymbol{v}$ component. Therefore the Aubin-Nitsche duality argument can only be applied to the approximation property involving $\boldsymbol{v}$.

By including h_k^2 with $\|q\|^2_{L_2(\Omega)}$, the approximation property involving q becomes trivial.

Multigrid Methods for Saddle Point Problems

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For the Darcy problem, the elliptic regularity result holds for the q component and hence we include h_k^2 with $\|\boldsymbol{v}\|^2_{L_2(\Omega)}$.

Multigrid Methods for Saddle Point Problems

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We can define these mesh-dependent inner products by mass lumping so that the matrices representing them with respect to the standard DOFs are diagonal matrices.