Multigrid Methods for Saddle Point Problems

Given $(\boldsymbol{f}, g) \in V_k \times Q_k$, we want to construct multigrid methods for

$$(*) \qquad B_k(\boldsymbol{v}, q) = (\boldsymbol{f}, g)$$

- Apply pre-smoothing steps with initial guess $(\boldsymbol{v}_0, q_0)$ to obtain an approximate solution $(\boldsymbol{v}_\dagger, q_\dagger)$ of $(*)$.
- Transfer the residual of $(\boldsymbol{v}_\dagger, q_\dagger)$ to a coarse grid and apply the multigrid algorithm on the coarse grid to find an approximate correction of $(\boldsymbol{v}_\dagger, q_\dagger)$.
- Apply post-smoothing steps to the corrected approximate solution of $(*)$ to obtain the final output.

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Two Ingredients

- Intergrid transfer operators
- Efficient smoother

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Coarse-to-Fine Operator

$I_{k-1}^{k} : V_{k-1} \times Q_{k-1} \longrightarrow V_k \times Q_k$ is the natural injection.

(The finite element spaces are nested.)

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Coarse-to-Fine Operator

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Fine-to-Coarse Operator

$I_k^{k-1} : V_k \times Q_k \longrightarrow V_{k-1} \times Q_{k-1}$ is the transpose of I_{k-1}^{k} with respect to the mesh-dependent inner product:

$$\big[I_k^{k-1}(\boldsymbol{v}, q), (\boldsymbol{w}, r)\big]_{k-1} = \big[(\boldsymbol{v}, q), I_{k-1}^{k}(\boldsymbol{w}, r)\big]_k$$

for all $(\boldsymbol{v}, q) \in V_k \times Q_k$ and $(\boldsymbol{w}, r) \in V_{k-1} \times Q_{k-1}$

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Coarse-to-Fine Operator

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Ritz Projection Operator

The operator $P_k^{k-1} : V_k \times Q_k \longrightarrow V_{k-1} \times Q_{k-1}$ is the transpose of I_{k-1}^k with respect to the variational bilinear form $\mathcal{B}(\cdot, \cdot)$.

$$\mathcal{B}\big(P_k^{k-1}(\boldsymbol{v}, q), (\boldsymbol{w}, r)\big) = \mathcal{B}\big((\boldsymbol{v}, q), I_{k-1}^k(\boldsymbol{w}, r)\big)$$

$$\forall\, (\boldsymbol{v}, q) \in V_k \times Q_k,\ (\boldsymbol{w}, r) \in V_{k-1} \times Q_{k-1}$$

Multigrid Methods for Saddle Point Problems

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$$(*) \qquad B_k(\boldsymbol{v}, q) = (\boldsymbol{f}, g)$$

Smoothing Step for $(*)$

$$(\boldsymbol{v}_{\mathrm{new}}, q_{\mathrm{new}}) = (\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}}) + S_k\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}})\big)$$

$S_k : V_k \times Q_k \longrightarrow V_k \times Q_k$ is a smoother.

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$S_k : V_k \times Q_k \longrightarrow V_k \times Q_k$ is a smoother.

Question How do we choose S_k?

Multigrid Methods for Saddle Point Problems

Given $(\boldsymbol{f}, g) \in V_k \times Q_k$, we want to construct multigrid methods for

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- $S_k B_k$ should be related to a scale of mesh-dependent norms on $V_k \times Q_k$ so that we can prove smoothing properties.

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- $S_k B_k$ should be related to a scale of mesh-dependent norms on $V_k \times Q_k$ so that we can prove smoothing properties.
- The scale of mesh-dependent norms should be related to a scale of Sobolev norms so that we can prove approximation properties without using H^2 regularity.

Smoother for Post-Smoothing

$$(\boldsymbol{v}_{\mathrm{new}}, q_{\mathrm{new}}) = (\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}}) + S_k\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}})\big)$$

Smoother for Post-Smoothing

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$$S_k = \gamma_k B_k \mathfrak{C}_k^{-1}$$

where the operator $\mathfrak{C}_k : V_k \times Q_k \longrightarrow V_k \times Q_k$ is SPD with respect to the mesh-dependent inner product $[\cdot,\cdot]_k$ and satisfies

Stokes

$$\big[\mathfrak{C}_k(\boldsymbol{v}, q), (\boldsymbol{v}, q)\big]_k \approx \|(\boldsymbol{v}, q)\|_E^2 = |\boldsymbol{v}|_{H^1(\Omega)}^2 + \|q\|_{L_2(\Omega)}^2$$

Darcy

$$\big[\mathfrak{C}_k(\boldsymbol{v}, q), (\boldsymbol{v}, q)\big]_k \approx \|(\boldsymbol{v}, q)\|_E^2 = \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + \|q\|_{H^1(\Omega;\mathcal{T}_h)}^2$$

$\gamma_k =$ (constant) h_k^2 is a damping factor.

Smoother for Post-Smoothing

$$(\boldsymbol{v}_{\text{new}}, q_{\text{new}}) = (\boldsymbol{v}_{\text{old}}, q_{\text{old}}) + S_k\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\text{old}}, q_{\text{old}})\big)$$

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We can define

$$\mathfrak{C}_k^{-1}(\boldsymbol{v}, q) = (L_k^{-1}\boldsymbol{v}, h_k^{-2} q)$$

where L_k^{-1} is an optimal preconditioner (DD or MG) of the discrete Laplace operator associated with V_k.

Smoother for Post-Smoothing

$$(\boldsymbol{v}_{\mathrm{new}}, q_{\mathrm{new}}) = (\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}}) + S_k\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}})\big)$$

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where the operator $\mathfrak{C}_k : V_k \times Q_k \longrightarrow V_k \times Q_k$ is SPD with respect to the mesh-dependent inner product $[\cdot, \cdot]_k$ and satisfies

Darcy

$$\big[\mathfrak{C}_k(\boldsymbol{v}, q), (\boldsymbol{v}, q)\big]_k \approx \|(\boldsymbol{v}, q)\|_E^2 = \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + \|q\|_{H^1(\Omega;\mathcal{T}_h)}^2$$

We can define

$$\mathfrak{C}_k^{-1}(\boldsymbol{v}, q) = (h_k^{-2}\boldsymbol{v}, L_k^{-1}q)$$

where L_k^{-1} is an optimal preconditioner (DD or MG) of the discrete DG Laplace operator associated with Q_k.

Smoother for Post-Smoothing

$$(\boldsymbol{v}_{\text{new}}, q_{\text{new}}) = (\boldsymbol{v}_{\text{old}}, q_{\text{old}}) + S_k\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\text{old}}, q_{\text{old}})\big)$$

$$S_k = \gamma_k B_k \mathfrak{C}_k^{-1}$$

Note that

$$(\boldsymbol{v}_{\text{new}}, q_{\text{new}}) = (\boldsymbol{v}_{\text{old}}, q_{\text{old}}) + \gamma_k B_k \mathfrak{C}_k^{-1}\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\text{old}}, q_{\text{old}})\big)$$

is just Richardson relaxation for the equivalent SPD system

$$B_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v}, q) = B_k \mathfrak{C}_k^{-1}(\boldsymbol{f}, g)$$

Smoother for Post-Smoothing

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$$B_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v}, q) = B_k \mathfrak{C}_k^{-1}(\boldsymbol{f}, g)$$

The error propagation operator $R_k : V_k \times Q_k \longrightarrow V_k \times Q_k$ for one post-smoothing step is given by

$$R_k = Id_k - \gamma_k B_k \mathfrak{C}_k^{-1} B_k$$

where Id_k is the identity operator on $V_k \times Q_k$.

Smoother for Post-Smoothing

$$(\boldsymbol{v}_{\mathrm{new}}, q_{\mathrm{new}}) = (\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}}) + S_k\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}})\big)$$

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$$(\boldsymbol{v}_{\mathrm{new}}, q_{\mathrm{new}}) = (\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}}) + \gamma_k B_k \mathfrak{C}_k^{-1}\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}})\big)$$

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$$R_k = Id_k - \gamma_k B_k \mathfrak{C}_k^{-1} B_k$$

The operator $B_k \mathfrak{C}_k^{-1} B_k : V_k \times Q_k \longrightarrow V_k \times Q_k$ plays a key role.

Properties of $B_k \mathfrak{C}_k^{-1} B_k$

Properties of $B_k\mathfrak{C}_k^{-1}B_k$

$$\left[B_k\mathfrak{C}_k^{-1}B_k(\boldsymbol{v},q),(\boldsymbol{v},q)\right]_k \approx \|(\boldsymbol{v},q)\|_E^2 \qquad \forall(\boldsymbol{v},q)\in V_k\times Q_k$$

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$$\begin{aligned} &\big[B_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v}, q), (\boldsymbol{v}, q)\big]_k^{1/2} \\ &\qquad = \big[\mathfrak{C}_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v}, q), \mathfrak{C}_k^{-1} B_k(\boldsymbol{v}, q)\big]_k^{1/2} \end{aligned}$$

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$$\begin{aligned}
&\big[B_k\mathfrak{C}_k^{-1}B_k(\boldsymbol{v},q),(\boldsymbol{v},q)\big]_k^{1/2}\\
&\qquad = \big[\mathfrak{C}_k\mathfrak{C}_k^{-1}B_k(\boldsymbol{v},q),\mathfrak{C}_k^{-1}B_k(\boldsymbol{v},q)\big]_k^{1/2}\\
&\qquad = \sup_{(\boldsymbol{w},r)\in V_k\times Q_k}\frac{\big[\mathfrak{C}_k\mathfrak{C}_k^{-1}B_k(\boldsymbol{v},q),(\boldsymbol{w},r)\big]_k}{\big[\mathfrak{C}_k(\boldsymbol{w},r),(\boldsymbol{w},r)\big]_k^{1/2}}
\end{aligned}$$

duality with respect to the inner product $\big[\mathfrak{C}_k\cdot,\cdot\big]_k$

Properties of $B_k \mathfrak{C}_k^{-1} B_k$

$$\big[B_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v},q),(\boldsymbol{v},q)\big]_k \approx \|(\boldsymbol{v},q)\|_E^2 \qquad \forall (\boldsymbol{v},q) \in V_k \times Q_k$$

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&\qquad = \sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\big[\mathfrak{C}_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v},q),(\boldsymbol{w},r)\big]_k}{\big[\mathfrak{C}_k(\boldsymbol{w},r),(\boldsymbol{w},r)\big]_k^{1/2}} \\
&\qquad \approx \sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v},q),(\boldsymbol{w},r)\big)}{\|(\boldsymbol{w},r)\|_E}
\end{aligned}$$

$$\big[B_k(\boldsymbol{v},q),(\boldsymbol{w},r)\big]_k = \mathcal{B}\big((\boldsymbol{v},q),(\boldsymbol{w},r)\big)$$

$$\big[\mathfrak{C}_k(\boldsymbol{w},r),(\boldsymbol{w},r)\big]_k \approx \|(\boldsymbol{w},r)\|_E^2$$

Properties of $B_k \mathfrak{C}_k^{-1} B_k$

$$\left[B_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v}, q), (\boldsymbol{v}, q)\right]_k \approx \|(\boldsymbol{v}, q)\|_E^2 \qquad \forall (\boldsymbol{v}, q) \in V_k \times Q_k$$

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&\qquad = \sup_{(\boldsymbol{w}, r) \in V_k \times Q_k} \frac{\left[\mathfrak{C}_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v}, q), (\boldsymbol{w}, r)\right]_k}{\left[\mathfrak{C}_k(\boldsymbol{w}, r), (\boldsymbol{w}, r)\right]_k^{1/2}} \\
&\qquad \approx \sup_{(\boldsymbol{w}, r) \in V_k \times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{\|(\boldsymbol{w}, r)\|_E} \\
&\qquad \approx \|(\boldsymbol{v}, q)\|_E
\end{aligned}$$

stability estimate for $\mathcal{B}(\cdot, \cdot)$

Properties of $B_k \mathfrak{C}_k^{-1} B_k$

Stokes

$$\big[B_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v},q),(\boldsymbol{v},q)\big]_k \approx \|(\boldsymbol{v},q)\|_E^2 = |\boldsymbol{v}|_{H^1(\Omega)}^2 + \|q\|_{L_2(\Omega)}^2$$

$$\big[(\boldsymbol{v},q),(\boldsymbol{v},q)\big]_k = \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + h_k^2 \|q\|_{L_2(\Omega)}^2$$

Darcy

$$\big[B_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v},q),(\boldsymbol{v},q)\big]_k \approx \|(\boldsymbol{v},q)\|_E^2 = \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + \|q\|_{H^1(\Omega;\mathcal{T}_h)}^2$$

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$$\left[B_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v}, q), (\boldsymbol{v}, q)\right]_k \approx \|(\boldsymbol{v}, q)\|_E^2 = \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + \|q\|_{H^1(\Omega;\mathcal{T}_h)}^2$$

$$\left[(\boldsymbol{v}, q), (\boldsymbol{v}, q)\right]_k = h_k^2 \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + \|q\|_{L_2(\Omega)}^2$$

It follows from standard inverse estimates that

spectral radius of $B_k \mathfrak{C}_k^{-1} B_k \leq C h_k^{-2}$

Properties of $B_k \mathfrak{C}_k^{-1} B_k$

We can choose the damping factor

$$\gamma_k = (\text{constant})\, h_k^2$$

so that the spectral radius of the damped operator $\gamma_k B_k \mathfrak{C}_k^{-1} B_k$ that appears in the error propagation operator

$$R_k = Id_k - \gamma_k B_k \mathfrak{C}_k^{-1} B_k$$

of one post-smoothing step is ≤ 1.

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of one post-smoothing step is ≤ 1.

This is similar to standard smoothers for second order SPD problems and therefore R_k will have similar smoothing properties.

Mesh-Dependent Norms

- One of the norms in the scale of mesh-dependent norms related to $S_k B_k$ is equivalent to the energy norm so that we can prove contraction number estimates in the energy norm.

- The scale of mesh-dependent norms is also related to fractional order Sobolev norms so that we can prove an approximation property for norms other than the $L_2(\Omega)$ norm, for which the duality argument does not require H^2 regularity.

First Scale of Mesh-Dependent Norms

For $0 \le s \le 1$, we can use the SPD operator $B_k \mathfrak{C}_k^{-1} B_k$ to define a scale of mesh-dependent norms:

$$\|\!|(\boldsymbol{v}, q)\|\!|_{s,k} = \left[(B_k \mathfrak{C}_k^{-1} B_k)^s (\boldsymbol{v}, q), (\boldsymbol{v}, q)\right]_k^{\frac{1}{2}} \qquad \forall\, (\boldsymbol{v}, q) \in V_k \times Q_k$$

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Connections with Sobolev Norms

First Scale of Mesh-Dependent Norms

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$$|||(\boldsymbol{v}, q)|||_{s,k} = \big[(B_k \mathfrak{C}_k^{-1} B_k)^s (\boldsymbol{v}, q), (\boldsymbol{v}, q)\big]_k^{\frac{1}{2}} \qquad \forall\, (\boldsymbol{v}, q) \in V_k \times Q_k$$

Connections with Sobolev Norms

Stokes

$$|||(\boldsymbol{v}, q)|||_{0,k}^2 = \big[(\boldsymbol{v}, q), (\boldsymbol{v}, q)\big]_k \approx \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + h_k^2 \|q\|_{L_2(\Omega)}^2$$

$$\begin{aligned} |||(\boldsymbol{v}, q)|||_{1,k}^2 &= \big[B_k \mathfrak{C}_k^{-1} B_k (\boldsymbol{v}, q), (\boldsymbol{w}, r)\big]_k \\ &\approx \|(\boldsymbol{v}, q)\|_E^2 = |\boldsymbol{v}|_{H^1(\Omega)}^2 + \|q\|_{L_2(\Omega)}^2 \end{aligned}$$

$$|||(\boldsymbol{v}, q)|||_{1-\alpha,k} \approx \|\boldsymbol{v}\|_{H^{1-\alpha}(\Omega)} + h_k^\alpha \|q\|_{L_2(\Omega)}$$

($\alpha \in (\frac{1}{2}, 1]$ is the index of elliptic regularity.)

First Scale of Mesh-Dependent Norms

For $0 \le s \le 1$, we can use the SPD operator $B_k \mathfrak{C}_k^{-1} B_k$ to define a scale of mesh-dependent norms:

$$\|\!|(\boldsymbol{v}, q)|\!\|_{s,k} = \left[(B_k \mathfrak{C}_k^{-1} B_k)^s (\boldsymbol{v}, q), (\boldsymbol{v}, q)\right]_k^{\frac{1}{2}} \qquad \forall\, (\boldsymbol{v}, q) \in V_k \times Q_k$$

Connections with Sobolev Norms

Darcy

$$\|\!|(\boldsymbol{v}, q)|\!\|_{0,k}^2 = \left[(\boldsymbol{v}, q), (\boldsymbol{v}, q)\right]_k \approx h_k^2 \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + \|q\|_{L_2(\Omega)}^2$$

$$\begin{aligned} \|\!|(\boldsymbol{v}, q)|\!\|_{1,k}^2 &= \left[B_k \mathfrak{C}_k^{-1} B_k (\boldsymbol{v}, q), (\boldsymbol{w}, r)\right]_k \\ &\approx \|(\boldsymbol{v}, q)\|_E^2 = \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + \|q\|_{H^1(\Omega;\mathcal{T}_h)}^2 \end{aligned}$$

$$\|\!|(\boldsymbol{v}, q)|\!\|_{1-\alpha,k} \approx h_k^\alpha \|\boldsymbol{v}\|_{L_2(\Omega)} + \|q\|_{H^{1-\alpha}(\Omega)}$$

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First Scale of Mesh-Dependent Norms

For $0 \le s \le 1$, we can use the SPD operator $B_k \mathfrak{C}_k^{-1} B_k$ to define a scale of mesh-dependent norms:

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Connections with Sobolev Norms

Darcy

$$|||(\boldsymbol{v}, q)|||_{0,k}^2 = \left[(\boldsymbol{v}, q), (\boldsymbol{v}, q)\right]_k \approx h_k^2 \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + \|q\|_{L_2(\Omega)}^2$$

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$$|||(\boldsymbol{v}, q)|||_{1-\alpha,k} \approx h_k^\alpha \|\boldsymbol{v}\|_{L_2(\Omega)} + \|q\|_{H^{1-\alpha}(\Omega)}$$

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Smoother for Pre-Smoothing

$$(\boldsymbol{v}_{\text{new}}, q_{\text{new}}) = (\boldsymbol{v}_{\text{old}}, q_{\text{old}}) + S_k\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\text{old}}, q_{\text{old}})\big)$$

Smoother for Pre-Smoothing

$$(\boldsymbol{v}_{\mathrm{new}}, q_{\mathrm{new}}) = (\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}}) + S_k\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\mathrm{old}}, q_{\mathrm{old}})\big)$$

$$S_k = \gamma_k \mathfrak{C}_k^{-1} B_k$$

The error propagation operator $R_k^* : V_k \times Q_k \longrightarrow V_k \times Q_k$ for one pre-smoothing step is given by

$$R_k^* = Id_k - \gamma_k \mathfrak{C}_k^{-1} B_k^2$$

Smoother for Pre-Smoothing

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The error propagation operator $R_k^* : V_k \times Q_k \longrightarrow V_k \times Q_k$ for one pre-smoothing step is given by

$$R_k^* = Id_k - \gamma_k \mathfrak{C}_k^{-1} B_k^2$$

The choice of the pre-smoother is motivated by the relation

$$\mathcal{B}\big(R_k(\boldsymbol{v}, q), (\boldsymbol{w}, r)\big) = \mathcal{B}\big((\boldsymbol{v}, q), R_k^*(\boldsymbol{w}, r)\big)$$

for all $(\boldsymbol{v}, q), (\boldsymbol{w}, r) \in V_k \times Q_k$

Second Scale of Mesh-Dependent Norms

For $1 \le s \le 2$, we can define by duality another scale of mesh-dependent norms:

$$|\!|\!|(\boldsymbol{v}, q)|\!|\!|_{s,k}^{*} = \sup_{(\boldsymbol{w},r)\in V_k \times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{|\!|\!|(\boldsymbol{w}, r)|\!|\!|_{2-s,k}} \qquad \forall\, (\boldsymbol{v}, q) \in V_k \times Q_k$$

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Second Scale of Mesh-Dependent Norms

For $1 \le s \le 2$, we can define by duality another scale of mesh-dependent norms:

$$\|\!|(\boldsymbol{v}, q)|\!\|_{s,k}^{*} = \sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{\|\!|(\boldsymbol{w}, r)|\!\|_{2-s,k}} \qquad \forall\, (\boldsymbol{v}, q) \in V_k \times Q_k$$

The $\|\!|\cdot|\!\|_{1,k}^{*}$ norm is also equivalent to the energy norm.

$$\begin{aligned}
\|\!|(\boldsymbol{v}, q)|\!\|_{1,k}^{*} &= \sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{\|\!|(\boldsymbol{w}, r)|\!\|_{1,k}} \\
&\approx \sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{\|(\boldsymbol{w}, r)\|_{E}}
\end{aligned}$$

$$\|\!|(\boldsymbol{w}, r)|\!\|_{1,k} \approx \|(\boldsymbol{w}, r)\|_{E}$$

Second Scale of Mesh-Dependent Norms

For $1 \le s \le 2$, we can define by duality another scale of mesh-dependent norms:

$$|\!|\!|(\boldsymbol{v}, q)|\!|\!|_{s,k}^{*} = \sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{|\!|\!|(\boldsymbol{w}, r)|\!|\!|_{2-s,k}} \qquad \forall\,(\boldsymbol{v}, q) \in V_k \times Q_k$$

The $|\!|\!| \cdot |\!|\!|_{1,k}^{*}$ norm is also equivalent to the energy norm.

$$\begin{aligned}
|\!|\!|(\boldsymbol{v}, q)|\!|\!|_{1,k}^{*} &= \sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{|\!|\!|(\boldsymbol{w}, r)|\!|\!|_{1,k}} \\
&\approx \sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v}, q), (\boldsymbol{w}, r)\big)}{\|(\boldsymbol{w}, r)\|_{E}} \\
&\approx \|(\boldsymbol{v}, q)\|_{E}
\end{aligned}$$

stability estimate for $\mathcal{B}(\cdot, \cdot)$

Convergence Analysis

Smoothing Properties

Post-Smoothing (first scale)

$$|||(R_k)^m(\boldsymbol{v},q)|||_{1,k} \leq C(h_k\sqrt{m})^{-\alpha}|||(\boldsymbol{v},q)|||_{1-\alpha,k}$$

- Spectral Theorem
- Calculus

$$|||(\boldsymbol{v},q)|||_{s,k} = \left[(B_k\mathfrak{C}_k^{-1}B_k)^s(\boldsymbol{v},q),(\boldsymbol{v},q)\right]_k^{\frac{1}{2}}$$

$$R_k = Id_k - \gamma_k B_k \mathfrak{C}_k^{-1} B_k$$

Smoothing Properties

Post-Smoothing (first scale)

$$\|\!|(R_k)^m(\boldsymbol{v},q)\|\!|_{1,k} \leq C(h_k\sqrt{m})^{-\alpha}\|\!|(\boldsymbol{v},q)\|\!|_{1-\alpha,k}$$

Pre-Smoothing (second scale)

$$\|\!|(R_k^*)^m(\boldsymbol{v},q)\|\!|^*_{1+\alpha,k} \leq C(h_k\sqrt{m})^{-\alpha}\|\!|(\boldsymbol{v},q)\|\!|^*_{1,k}$$

- duality

Approximation Properties

Approximation Property (first scale)

$$|||(Id_k - I_{k-1}^k P_k^{k-1})(\boldsymbol{v}, q)|||_{1-\alpha,k} \leq C h_k^\alpha |||(\boldsymbol{v}, q)|||_{1,k}$$

- elliptic regularity
- Aubin-Nitsche duality argument
- properly weighted mesh dependent inner product

Approximation Properties

Approximation Property (first scale)

$$|||(Id_k - I_{k-1}^k P_k^{k-1})(\boldsymbol{v}, q)|||_{1-\alpha,k} \leq C h_k^\alpha |||(\boldsymbol{v}, q)|||_{1,k}$$

Approximation Property (second scale)

$$|||(Id_k - I_{k-1}^k P_k^{k-1})(\boldsymbol{v}, q)|||_{1,k}^* \leq C h_k^\alpha |||(\boldsymbol{v}, q)|||_{1+\alpha,k}^*$$

- duality

Convergence Results

The smoothing and approximation properties lead to the uniform convergence of the two-grid algorithm and hence the W-cycle algorithm provided the number of smoothing steps (independent of mesh size) is sufficiently large.

Asymptotically

$$\text{contraction number} \leq Cm^{-\alpha}$$

C is independent of h.

m is the number of pre-smoothing and post-smoothing steps

α is the index of elliptic regularity. ($\alpha = 1$ for convex Ω)

Nonsymmetric Saddle Point Problems

The results for symmetric saddle point problems can be extended to nonsymmetric saddle point problems provided we have the following stability estimates:

$$\sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{v},q),(\boldsymbol{w},r)\big)}{\|(\boldsymbol{w},r)\|_E} \approx \|(\boldsymbol{v},q)\|_E$$

$$\sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{w},r),(\boldsymbol{v},q)\big)}{\|(\boldsymbol{w},r)\|_E} \approx \|(\boldsymbol{v},q)\|_E$$

Oseen System

Darcy System with a convective term

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$$\sup_{(\boldsymbol{w},r)\in V_k\times Q_k} \frac{\mathcal{B}\big((\boldsymbol{w},r),(\boldsymbol{v},q)\big)}{\|(\boldsymbol{w},r)\|_E} \approx \|(\boldsymbol{v},q)\|_E$$

Oseen System

Darcy System with a convective term

It only requires a simple modification of the smoothers.

Nonsymmetric Saddle Point Problems

$$(\boldsymbol{v}_{\text{new}}, q_{\text{new}}) = (\boldsymbol{v}_{\text{old}}, q_{\text{old}}) + S_k\big((\boldsymbol{f}, g) - B_k(\boldsymbol{v}_{\text{old}}, q_{\text{old}})\big)$$

Post-Smoothing

$$S_k = \gamma_k B_k^t \mathfrak{C}_k^{-1} B_k \qquad (S_k = \gamma_k B_k \mathfrak{C}_k^{-1} B_k)$$

Pre-Smoothing

$$S_k = \gamma_k \mathfrak{C}_k^{-1} B_k^t B_k \qquad (S_k = \gamma_k \mathfrak{C}_k^{-1} B_k^2)$$

$B_k^t : V_k \times Q_k \longrightarrow V_k \times Q_k$ is the transpose of B_k with respect to the mesh-dependent inner product $[\cdot, \cdot]_k$.

$$\big[B_k(\boldsymbol{v}, q), (\boldsymbol{w}, r)\big]_k = \big[(\boldsymbol{v}, q), B_k^t(\boldsymbol{w}, r)\big]_k$$

for all $(\boldsymbol{v}, q), (\boldsymbol{w}, r) \in V_k \times Q_k$

Nonsymmetric Saddle Point Problems

The proofs of the smoothing properties remain the same.

The proofs of the approximation properties require duality arguments for both the saddle point problem and the adjoint problem.

V-Cycle Algorithm

Numerical results indicate that the V-cycle algorithm, where the $(k-1)$-st level iteration is applied once in the coarse grid correction step, is also uniformly convergent.

V-Cycle Algorithm

Numerical results indicate that the V-cycle algorithm, where the $(k-1)$-st level iteration is applied once in the coarse grid correction step, is also uniformly convergent.

Recall that the post-smoothing step is just Richardson relaxation applied to the SPD system

$$B_k \mathfrak{C}_k^{-1} B_k(\boldsymbol{v}, q) = (\boldsymbol{f}, g)$$

and the SPD operator behaves like a (nonstandard) second order differential operator.

It should be possible to establish results for the V-cycle algorithm by using techniques from the additive multigrid theory for nonstandard finite element methods.

B. 2002, 2004

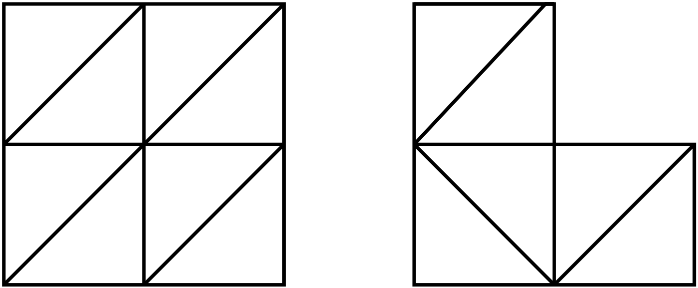
Numerical Results

Numerical Results

Stokes and Lamé

$V_k \times Q_k = P_2 – P_1$ Taylor-Hood finite element pair

A $V(2,2)$ multigrid Laplace solve is used in the construction of $\mathfrak{C}_k^{-1}(\boldsymbol{v}, q) = (L_k^{-1}\boldsymbol{v}, h_k^{-2}q)$.



Unit Square

L-Shaped

Numerical Results

Stokes System

W-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(1, 1)	0.82	0.86	0.86	0.86	0.86	0.86
(2, 2)	0.76	0.78	0.78	0.78	0.78	0.78
(4, 4)	0.66	0.68	0.69	0.69	0.69	0.69
(8, 8)	0.55	0.56	0.56	0.56	0.56	0.56
(16, 16)	0.38	0.39	0.39	0.39	0.39	0.39
(32, 32)	0.19	0.19	0.19	0.19	0.19	0.19

Contraction numbers for the unit square
in the energy norm $\|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}$

Numerical Results

Stokes System

W-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(1, 1)	0.88	0.90	0.90	0.90	0.90	0.90
(2, 2)	0.81	0.83	0.84	0.84	0.84	0.84
(4, 4)	0.73	0.75	0.75	0.75	0.75	0.75
(8, 8)	0.63	0.65	0.65	0.65	0.65	0.65
(16, 16)	0.48	0.49	0.49	0.49	0.49	0.49
(32, 32)	0.28	0.29	0.29	0.29	0.29	0.29

Contraction numbers for the L-shaped domain
in the energy norm $\|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}$

Numerical Results

Stokes System

V-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(1, 1)	0.87	0.90	0.90	0.90	0.90	0.90
(2, 2)	0.76	0.76	0.79	0.80	0.80	0.80
(4, 4)	0.66	0.70	0.70	0.70	0.70	0.70
(8, 8)	0.55	0.57	0.57	0.58	0.58	0.58
(16, 16)	0.38	0.40	0.40	0.40	0.40	0.40
(32, 32)	0.19	0.19	0.20	0.20	0.20	0.20

Contraction numbers for the unit square

in the energy norm $\|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}$

Numerical Results

Lamé System $(\mu = 1,\, \lambda = 500,\, \nu = 0.499)$

W-cycle algorithm

$(m_1, m_2)\backslash k$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
(1, 1)	0.92	0.93	0.93	0.93	0.93	0.93
(2, 2)	0.93	0.93	0.93	0.93	0.93	0.93
(4, 4)	0.88	0.89	0.90	0.90	0.90	0.90
(8, 8)	0.81	0.83	0.83	0.83	0.83	0.83
(16, 16)	0.70	0.72	0.72	0.72	0.72	0.72
(32, 32)	0.53	0.54	0.54	0.54	0.54	0.54
(64, 64)	0.31	0.32	0.32	0.32	0.32	0.32

Contraction numbers for the unit square
in the energy norm $\|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}$

Numerical Results

Lamé System $(\mu = 1, \lambda = 500, \nu = 0.499)$

W-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(1, 1)	0.94	0.95	0.95	0.95	0.95	0.95
(2, 2)	0.94	0.95	0.95	0.95	0.95	0.95
(4, 4)	0.91	0.92	0.92	0.92	0.92	0.92
(8, 8)	0.85	0.87	0.87	0.87	0.87	0.87
(16, 16)	0.76	0.79	0.79	0.79	0.79	0.79
(32, 32)	0.62	0.64	0.64	0.64	0.64	0.64
(64, 64)	0.43	0.44	0.44	0.44	0.44	0.44

Contraction numbers for the L-shaped domain

in the energy norm $\|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}$

Numerical Results

Lamé System $(\mu = 1, \lambda = 500, \nu = 0.499)$

V-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(1, 1)	0.92	0.94	0.94	0.94	0.94	0.94
(2, 2)	0.93	0.94	0.94	0.94	0.94	0.93
(4, 4)	0.88	0.90	0.90	0.90	0.90	0.90
(8, 8)	0.81	0.83	0.84	0.84	0.84	0.84
(16, 16)	0.70	0.73	0.73	0.73	0.73	0.73
(32, 32)	0.53	0.55	0.54	0.55	0.55	0.55
(64, 64)	0.31	0.32	0.32	0.32	0.32	0.32

Contraction numbers for the unit square
in the energy norm $\|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}$

Numerical Results

Lamé System $(\mu = 1, k = 5)$

W-cycle algorithm

$(m_1, m_2)\backslash\lambda$	$\lambda = 10^0$	$\lambda = 10^1$	$\lambda = 10^2$	$\lambda = 10^3$
(1, 1)	0.96	0.95	0.92	0.93
(2, 2)	0.93	0.93	0.92	0.93
(4, 4)	0.87	0.87	0.88	0.90
(8, 8)	0.76	0.79	0.81	0.83
(16, 16)	0.62	0.67	0.68	0.72
(32, 32)	0.48	0.47	0.49	0.55

Contraction numbers for the unit square
in the energy norm $\|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}$

Numerical Results

Oseen System

Find $(\boldsymbol{u}, p) \in [H_0^1(\Omega)]^2 \times L_2^0(\Omega)$ such that

$$
\begin{aligned}
a(\boldsymbol{u}, v) + b(\boldsymbol{v}, p) &= F(v) \qquad && \forall\, \boldsymbol{v} \in [H_0^1(\Omega)]^2 \\
b(\boldsymbol{u}, q) &= 0 && \forall\, q \in L_2^0(\Omega)
\end{aligned}
$$

$$
a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \left[(\nabla \boldsymbol{u} : \nabla \boldsymbol{v}) + (\boldsymbol{w} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{v}\right] dx
$$

$$
b(\boldsymbol{v}, p) = -\int_\Omega (\nabla \cdot \boldsymbol{v}) p \, dx
$$

where the wind function

$$
\boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Numerical Results

Oseen System

W-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(1, 1)	0.91	0.91	0.91	0.91	0.91	0.91
(2, 2)	0.84	0.83	0.83	0.83	0.83	0.83
(4, 4)	0.71	0.71	0.71	0.71	0.71	0.71
(8, 8)	0.59	0.61	0.61	0.61	0.61	0.61
(16, 16)	0.45	0.47	0.47	0.47	0.47	0.47
(32, 32)	0.28	0.29	0.29	0.29	0.29	0.29

Contraction numbers for the L-shaped domain
in the energy norm $\|\boldsymbol{v}\|_{H^1(\Omega)} + \|q\|_{L_2(\Omega)}$

Numerical Results

Darcy System

We take K to be the identity matrix.

$V_k \times Q_k = RT_1 \times P_1$ Raviart-Thomas finite element pair

A $V(4,4)$ multigrid DG Laplace solve is used in the construction of $\mathfrak{C}_k^{-1}(\boldsymbol{v}, q) = (h_k^{-2}\boldsymbol{v}, L_k^{-1}q)$.

Ω is either a unit square or an L-shaped domain.

Numerical Results

Darcy System

W-cycle algorithm

$(m_1, m_2)\backslash k$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
(10, 10)	0.80	0.81	0.81	0.81	0.81	0.81
(20, 20)	0.66	0.67	0.67	0.67	0.67	0.67
(40, 40)	0.47	0.48	0.48	0.48	0.48	0.48
(80, 80)	0.24	0.24	0.24	0.24	0.24	0.24

V-cycle algorithm

$(m_1, m_2)\backslash k$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
(10, 10)	0.80	0.82	0.81	0.82	0.82	0.82
(20, 20)	0.66	0.68	0.68	0.68	0.68	0.68
(40, 40)	0.47	0.48	0.48	0.48	0.48	0.48
(80, 80)	0.24	0.25	0.25	0.25	0.25	0.25

Contraction numbers for the unit square

in the energy norm $\|\boldsymbol{v}\|_{L_2(\Omega)} + \|q\|_{H^1(\Omega;\mathcal{T}_h)}$

Numerical Results

Darcy System

W-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(10, 10)	0.81	0.82	0.82	0.82	0.82	0.82
(20, 20)	0.70	0.70	0.70	0.70	0.70	0.70
(40, 40)	0.51	0.51	0.51	0.51	0.51	0.51
(80, 80)	0.28	0.28	0.28	0.28	0.28	0.28

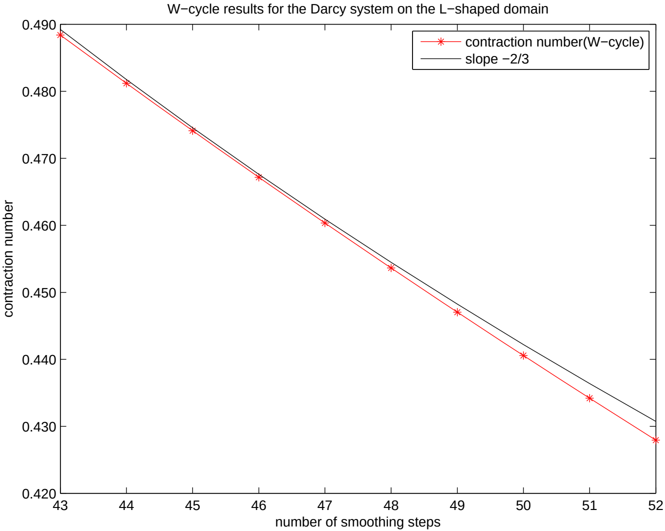
V-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(10, 10)	0.81	0.82	0.82	0.82	0.82	0.82
(20, 20)	0.70	0.70	0.70	0.70	0.70	0.70
(40, 40)	0.51	0.52	0.51	0.51	0.51	0.51
(80, 80)	0.28	0.28	0.28	0.28	0.28	0.28

Contraction numbers for the L-shaped domain

in the energy norm $\|\boldsymbol{v}\|_{L_2(\Omega)} + \|q\|_{H^1(\Omega;\mathcal{T}_h)}$

Numerical Results



Numerical Results

Darcy System with a Convective Term

Find $(\boldsymbol{u}_k, p_k) \in V_k \times Q_k$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{u}_k \cdot \boldsymbol{v}\,dx + \int_\Omega (\nabla \cdot \boldsymbol{v}) p_k\,dx &= 0 && \forall\,\boldsymbol{v} \in V_k \\ \int_\Omega (\nabla \cdot u_k) q\,dx - \int_\Omega (\boldsymbol{b} \cdot \nabla_{h_k} p_k) q\,dx &= G(q) && \forall\,q \in Q_k \end{aligned}$$

where

$$\boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Numerical Results

Darcy System with a Convective Term

W-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(10, 10)	0.80	0.81	0.81	0.81	0.81	0.81
(20, 20)	0.67	0.68	0.67	0.67	0.67	0.67
(40, 40)	0.48	0.48	0.48	0.48	0.48	0.48
(80, 80)	0.24	0.24	0.24	0.24	0.24	0.24

V-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(10, 10)	0.80	0.81	0.81	0.81	0.81	0.81
(20, 20)	0.67	0.68	0.67	0.67	0.67	0.67
(40, 40)	0.48	0.48	0.48	0.48	0.48	0.48
(80, 80)	0.24	0.24	0.24	0.24	0.24	0.24

Contraction numbers for the unit square

in the energy norm $\|\boldsymbol{v}\|_{L_2(\Omega)} + \|q\|_{H^1(\Omega;\mathcal{T}_h)}$

Numerical Results

Darcy System with a Convective Term

W-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(10, 10)	0.81	0.82	0.82	0.82	0.82	0.82
(20, 20)	0.70	0.70	0.70	0.70	0.70	0.70
(40, 40)	0.52	0.52	0.52	0.52	0.52	0.52
(80, 80)	0.29	0.29	0.29	0.29	0.29	0.29

V-cycle algorithm

$(m_1, m_2)\backslash k$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
(10, 10)	0.81	0.82	0.82	0.82	0.82	0.82
(20, 20)	0.70	0.70	0.70	0.70	0.70	0.70
(40, 40)	0.52	0.52	0.52	0.52	0.52	0.52
(80, 80)	0.29	0.29	0.29	0.30	0.30	0.30

Contraction numbers for the L-shaped domain

in the energy norm $\|\boldsymbol{v}\|_{L_2(\Omega)} + \|q\|_{H^1(\Omega;\mathcal{T}_h)}$

Concluding Remarks

Summary: A Recipe for Saddle Point Problems

Find $(u,p) \in V \times Q$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= F(v) \qquad \forall\, v \in V \\ b(u,q) - c(p,q) &= G(q) \qquad \forall\, q \in Q \end{aligned}$$

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- For a saddle point problem arising from a $2m$-order elliptic boundary value problem, we should use a (closed) subspace of the Sobolev space $H^m(\Omega)$ for the unknown that comes with elliptic regularity estimates, and a (closed) subspace of $L_2(\Omega)$ for the other component.

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- For saddle point problems of the first type, u is the unknown associated with $H^m(\Omega)$. In this case one can use stable conforming mixed finite element spaces $V_k \times Q_k$ and the mesh-dependent inner product should satisfy

$$[(v,q),(v,q)]_k \approx \|v\|^2_{L_2(\Omega)} + h_k^{2m}\|q\|^2_{L_2(\Omega)} \qquad \forall\,(v,q) \in V_k \times Q_k$$

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- For saddle point problems of the second type, p is the unknown associated with $H^m(\Omega)$. In this case one can treat stable conforming mixed finite element spaces $V_k \times Q_k$ for the (more popular) dual formulation as nonconforming mixed finite element spaces for the saddle point problem, and the mesh-dependent inner product should satisfy

$$[(v, q), (v, q)]_k \approx h_k^{2m} \|v\|^2_{L_2(\Omega)} + \|q\|^2_{L_2(\Omega)}$$

for all $(v, q) \in V_k \times Q_k$ (The finite element space Q_k should contain polynomials of degree $\leq m$.)

Summary: A Recipe for Saddle Point Problems

- For both types of saddle point problems the key is the construction of a preconditioner $\mathfrak{C}_k^{-1}$ such that

$$\big[\mathfrak{C}_k(v,q),(v,q)\big]_k \approx \|(v,q)\|_E^2 \qquad \forall\,(v,q) \in V_k \times Q_k$$

$$\|(v,q)\|_E = \|v\|_{H^m(\Omega)} + \|q\|_{L_2(\Omega)} \qquad \text{(type-I)}$$

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- For type-I problems, this involves an optimal preconditioner for the discrete elliptic operator of order $2m$ associated with the conforming space V_k. For type-II problems, this involves an optimal preconditioner for the discrete elliptic operator of order $2m$ associated with the nonconforming space Q_k, where a DG discretization appears naturally.

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- One can then develop uniformly convergent multigrid algorithms in the energy norm for general polyhedral domains.

Other Examples

Other Examples

- Saddle point problems for linear elasticity in the stress-displacement formulation are examples of type-II problems.

  Stability analysis with respect to mesh-dependent norms was carried out in Stenberg 1988 for mixed finite element methods where the symmetry of the stress is weakly enforced.

Other Examples

- Saddle point problems for the biharmonic equation are also examples of type-II problems.

  Stability analysis of mixed finite element methods with respect to mesh-dependent norms was carried out in Babuška-Osborn-Pitkäranta 1980.

  The DG methods for the construction of optimal preconditioners are precisely the C^0 interior penalty methods for fourth order problems.

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