hp *DG*, *CG* and *QTT* FEM

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Outline

1. Analytic Regularity of Elliptic Problems in Polyhedra
2. Exponential Convergence of hp-DG FEM
3. Numerical Experiments
4. Exponential Convergence of hp-CG FEM
5. Reduced Basis Methods (RBM) and Model Order Reduction (MOR)
6. Quantized Tensor Train (QTT) FEM
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Analytic Regularity of Elliptic Problems in Polyhedra

- $\Omega \subset \mathbb{R}^3$ open polyhedron, plane faces $\Gamma_{\boldsymbol{\iota}}$, $\boldsymbol{\iota} \in \mathcal{J} = \mathcal{J}_D \,\dot{\cup}\, \mathcal{J}_N$,
- $\Gamma = \partial\Omega = \dot{\bigcup}_{\boldsymbol{\iota}\in\mathcal{J}} \overline{\Gamma_{\boldsymbol{\iota}}}$,
- $\boldsymbol{A}$ constant, symmetric, positive definite, f analytic in $\overline{\Omega}$.

$$
\begin{aligned}
-\nabla \cdot (\boldsymbol{A}\nabla)u &= f && \text{in } \Omega \subset \mathbb{R}^3, \\
\gamma_0(u) &= 0 && \text{on } \Gamma_{\boldsymbol{\iota}} \subset \partial\Omega, \quad \boldsymbol{\iota} \in \mathcal{J}_D, \\
\gamma_1(u) &= 0 && \text{on } \Gamma_{\boldsymbol{\iota}} \subset \partial\Omega, \quad \boldsymbol{\iota} \in \mathcal{J}_N \ .
\end{aligned}
$$

Variational Form: $V := \{v \in H^1(\Omega) : v|_{\Gamma_{\boldsymbol{\iota}}} = 0,\ \boldsymbol{\iota} \in \mathcal{J}_D\}$.

$$
u \in V : \quad a(u,v) := \int_\Omega \boldsymbol{A}\nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} = \int_\Omega f v \,\mathrm{d}\boldsymbol{x} \qquad \forall v \in V.
$$

Lax-Milgram: $\forall f \in V^* \exists! u \in V$; $V = H^1(\Omega)/\mathbb{R}$ if $\mathcal{J}_D = \emptyset$, $\int_\Omega f \,\mathrm{d}\boldsymbol{x} = 0$.

Analytic Regularity of Elliptic Problems in Polyhedra

Proposition [Agmon, Douglis, Nirenberg 1964][C.B. Morrey 1966]

- If $f \in A(\Omega)$, then for every $x \in \Omega$, u is analytic at x.
- u admits unique analytic continuation to $\Gamma \backslash \mathcal{S}$, where
- singular set $\mathcal{S} \subset \partial\Omega$ is

$$\mathcal{S} := \mathcal{C} \,\dot{\cup}\, \mathcal{E} = \Big(\bigcup_{\boldsymbol{c} \in \mathcal{C}} \boldsymbol{c} \Big) \,\dot{\cup}\, \Big(\bigcup_{\boldsymbol{e} \in \mathcal{E}} \boldsymbol{e} \Big) \subset \Gamma \,, \tag{1}$$

$\mathcal{C}$: finite set of corners $\boldsymbol{c}$, $\mathcal{E}$: finite set of (open) edges $\boldsymbol{e}$ of Ω.

Analogous results for general elliptic systems w. nonconstant (analytic in $\overline{\Omega}$) coefficients, nonlinear problems,...[C.B. Morrey Jr. 1966]

Analytic Regularity of Elliptic Problems in Polyhedra

Exponential Convergence requires **quantification of analyticity**:

- **Finite Order Weighted Sobolev and Hölder Spaces**:
  V.A. Kondrat'ev (Conical Domains 1967), Nikish'kin (Polyhedra 1976),
  Maz'ya & B.A. Plamenevskii (Polyhedra 1983), Grisvard (1988-),
  Maz'ya & J. Rossmann, AMS (2012),
- **Countably Normed Weighted Sobolev Spaces**:
  Bolley, Camus, Dauge (Conical 2d, 1979),
  Babuška, Guo (1986-1988) (Ω Polygon 2d), (1997) (Ω Polyhedron Ω 3d),
  Melenk (2002) (Singular Perturbations, Polygon Ω 2d),
- **Analytic Regularity Shifts in Polyhedra**:
  Costabel, Dauge & Nicaise (Polyhedra 3d) (M3AS 2012).

Analytic Regularity of Elliptic Problems in Polyhedra

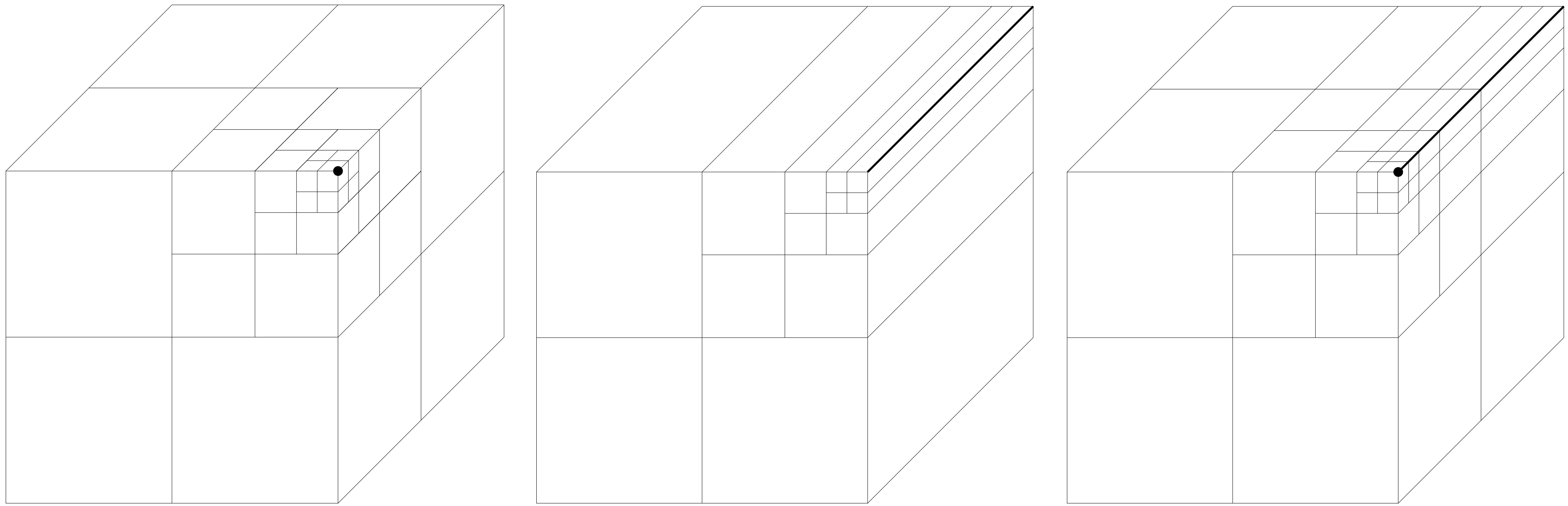


Figure 1: Three basic geometric mesh subdivisions in reference patch $\widetilde{Q}$ with subdivision ratio $\sigma = \frac{1}{2}$:
isotropic towards corner c (left),
anisotropic towards edge e (center), and
anisotropic towards one corner-edge pair ce (right).

Analytic Regularity of Elliptic Problems in Polyhedra

Subdomains and Weights:

For $\boldsymbol{c} \in \mathcal{C}$, $\boldsymbol{e} \in \mathcal{E}$, and $\boldsymbol{x} \in \Omega$, define distance functions

$$r_{\boldsymbol{c}}(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{c}|, \qquad r_{\boldsymbol{e}}(\boldsymbol{x}) = \inf_{\boldsymbol{y} \in \boldsymbol{e}} |\boldsymbol{x} - \boldsymbol{y}|, \qquad \rho_{\boldsymbol{ce}}(\boldsymbol{x}) = r_{\boldsymbol{e}}(\boldsymbol{x}) / r_{\boldsymbol{c}}(\boldsymbol{x}).$$

For each corner $\boldsymbol{c} \in \mathcal{C}$, denote $\mathcal{E}_{\boldsymbol{c}} := \{ \boldsymbol{e} \in \mathcal{E} \,:\, \boldsymbol{c} \cap \overline{\boldsymbol{e}} \neq \emptyset \}$ set of all edges of Ω which meet at $\boldsymbol{c}$.
For each edge $\boldsymbol{e} \in \mathcal{E}$, set of corners of $\boldsymbol{e}$ is $\mathcal{C}_{\boldsymbol{e}} := \{ \boldsymbol{c} \in \mathcal{C} \,:\, \boldsymbol{c} \cap \overline{\boldsymbol{e}} \neq \emptyset \}$.
For $\varepsilon > 0$ small, $\boldsymbol{c} \in \mathcal{C}$, $\boldsymbol{e} \in \mathcal{E}$ respectively $\boldsymbol{e} \in \mathcal{E}_{\boldsymbol{c}}$, define disjoint **neighborhoods**

$$\begin{aligned}
\omega_{\boldsymbol{c}} &= \{ \boldsymbol{x} \in \Omega \,:\, r_{\boldsymbol{c}}(\boldsymbol{x}) < \varepsilon \ \wedge \ \rho_{\boldsymbol{ce}}(\boldsymbol{x}) > \varepsilon \quad \forall\, \boldsymbol{e} \in \mathcal{E}_{\boldsymbol{c}} \}, \\
\omega_{\boldsymbol{e}} &= \{ \boldsymbol{x} \in \Omega \,:\, r_{\boldsymbol{e}}(\boldsymbol{x}) < \varepsilon \ \wedge \ r_{\boldsymbol{c}}(\boldsymbol{x}) > \varepsilon \quad \forall\, \boldsymbol{c} \in \mathcal{C}_{\boldsymbol{e}} \}, \\
\omega_{\boldsymbol{ce}} &= \{ \boldsymbol{x} \in \Omega \,:\, r_{\boldsymbol{c}}(\boldsymbol{x}) < \varepsilon \ \wedge \ \rho_{\boldsymbol{ce}}(\boldsymbol{x}) < \varepsilon \}.
\end{aligned}$$

Corner, *Edge* and *Corner-Edge* and *Interior* neighborhoods of Ω:

$$\Omega_{\mathcal{C}} = \bigcup_{\boldsymbol{c} \in \mathcal{C}} \omega_{\boldsymbol{c}}, \qquad \Omega_{\mathcal{E}} = \bigcup_{\boldsymbol{e} \in \mathcal{E}} \omega_{\boldsymbol{e}}, \qquad \Omega_{\mathcal{CE}} = \bigcup_{\boldsymbol{c} \in \mathcal{C}} \bigcup_{\boldsymbol{e} \in \mathcal{E}_{\boldsymbol{c}}} \omega_{\boldsymbol{ce}}, \qquad \Omega_{int} := \Omega \setminus \overline{\Omega_{\mathcal{C}} \cup \Omega_{\mathcal{E}} \cup \Omega_{\mathcal{CE}}}\ .$$

Analytic Regularity of Elliptic Problems in Polyhedra

Weighed Sobolev Spaces I:

- Corner and Edge exponents: $\beta_{\boldsymbol{c}}, \beta_{\boldsymbol{e}} \in \mathbb{R}$, $-2 < \beta_{\boldsymbol{c}}, \beta_{\boldsymbol{e}} < -1$
- Weight exponent vector: $\boldsymbol{\beta} = \{\beta_{\boldsymbol{c}} : \boldsymbol{c} \in \mathcal{C}\} \cup \{\beta_{\boldsymbol{e}} : \boldsymbol{e} \in \mathcal{E}\} \in]-2,-1[^{|\mathcal{C}|+|\mathcal{E}|}$
- Local coordinate systems in $\omega_{\boldsymbol{e}}$ and $\omega_{\boldsymbol{ce}}$:
  for $\boldsymbol{c} \in \mathcal{C}$ and $\boldsymbol{e} \in \mathcal{E}_{\boldsymbol{c}}$, such that edge $\boldsymbol{e}$ corresponds to the direction $(0,0,1)$.
- Quantities transversal to $\boldsymbol{e}$: $(\cdot)^{\perp}$, Quantities parallel to $\boldsymbol{e}$: $(\cdot)^{\|}$.
  e.g. $\boldsymbol{x}^{\|}$ and $\boldsymbol{x}^{\perp}$
- If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ is multi-index of order $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \alpha_3$,
  then $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{\perp}, \alpha^{\|})$ with $\boldsymbol{\alpha}^{\perp} = (\alpha_1, \alpha_2)$ and $\alpha^{\|} = \alpha_3$,
- Edge-parallel $\|$ and $\perp$ derivatives $\mathsf{D}^{\boldsymbol{\alpha}}$: $\mathsf{D}^{\boldsymbol{\alpha}} = \mathsf{D}_{\perp}^{\boldsymbol{\alpha}^{\perp}} \mathsf{D}_{\|}^{\alpha^{\|}}$, where in $\mathsf{D}_{\perp}^{\boldsymbol{\alpha}^{\perp}}$ and $\mathsf{D}_{\|}^{\alpha^{\|}}$
- $\mathsf{D}_{\perp}$: gradient in edge-perpendicular direction, and $\mathsf{D}_{\|} = \mathsf{D}_{\|}^{1}$.
- $h_K^{||}$, $h_K^{\perp}$, $\boldsymbol{p}_K^{||}$, $\boldsymbol{p}_K^{\perp}$.

Analytic Regularity of Elliptic Problems in Polyhedra

Weighted Sobolev Spaces II: Seminorms $|u|_{N^k_{\boldsymbol{\beta}}(\Omega;\mathcal{C}',\mathcal{E}')}$ for $k \geq 0$, $\emptyset \subseteq \mathcal{C}' \subseteq \mathcal{C}$ and $\emptyset \subseteq \mathcal{E}' \subseteq \mathcal{E}$.

$$
\begin{aligned}
|u|^2_{N^k_{\boldsymbol{\beta}}(\Omega;\mathcal{C}',\mathcal{E}')} := \sum_{|\boldsymbol{\alpha}|=k} \Bigg\{ & \|\mathsf{D}^{\boldsymbol{\alpha}} u\|^2_{L^2(\Omega_{int})} \\
& + \sum_{\boldsymbol{c}\in\mathcal{C}'} \left\| r_{\boldsymbol{c}}^{\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{c}})} + \sum_{\boldsymbol{c}\in\mathcal{C}\setminus\mathcal{C}'} \left\| r_{\boldsymbol{c}}^{\max\{\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|,0\}} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{c}})} \\
& + \sum_{\boldsymbol{e}\in\mathcal{E}'} \left\| r_{\boldsymbol{e}}^{\beta_{\boldsymbol{e}}+\left|\boldsymbol{\alpha}^{\perp}\right|} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{e}})} + \sum_{\boldsymbol{e}\in\mathcal{E}\setminus\mathcal{E}'} \left\| r_{\boldsymbol{e}}^{\max\{\beta_{\boldsymbol{e}}+\left|\boldsymbol{\alpha}^{\perp}\right|,0\}} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{e}})} \\
& + \sum_{\boldsymbol{c}\in\mathcal{C}'} \sum_{\boldsymbol{e}\in\mathcal{E}_{\boldsymbol{c}}\cap\mathcal{E}'} \left\| r_{\boldsymbol{c}}^{\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|} \rho_{\boldsymbol{ce}}^{\beta_{\boldsymbol{e}}+\left|\boldsymbol{\alpha}^{\perp}\right|} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{ce}})} \\
& + \sum_{\boldsymbol{c}\in\mathcal{C}'} \sum_{\boldsymbol{e}\in\mathcal{E}_{\boldsymbol{c}}\cap(\mathcal{E}\setminus\mathcal{E}')} \left\| r_{\boldsymbol{c}}^{\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|} \rho_{\boldsymbol{ce}}^{\max\{\beta_{\boldsymbol{e}}+\left|\boldsymbol{\alpha}^{\perp}\right|,0\}} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{ce}})} \\
& + \sum_{\boldsymbol{c}\in\mathcal{C}\setminus C'} \sum_{\boldsymbol{e}\in\mathcal{E}_{\boldsymbol{c}}\cap\mathcal{E}'} \left\| r_{\boldsymbol{c}}^{\max\{\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|,0\}} \rho_{\boldsymbol{ce}}^{\beta_{\boldsymbol{e}}+\left|\boldsymbol{\alpha}^{\perp}\right|} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{ce}})} \\
& + \sum_{\boldsymbol{c}\in\mathcal{C}\setminus\mathcal{C}'} \sum_{\boldsymbol{e}\in\mathcal{E}_{\boldsymbol{c}}\cap(\mathcal{E}\setminus\mathcal{E}')} \left\| r_{\boldsymbol{c}}^{\max\{\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|,0\}} \rho_{\boldsymbol{ce}}^{\max\{\beta_{\boldsymbol{e}}+\left|\boldsymbol{\alpha}^{\perp}\right|,0\}} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{ce}})} \Bigg\} \,.
\end{aligned}
$$

Analytic Regularity of Elliptic Problems in Polyhedra

Weighted Sobolev Spaces III: Norms $\|u\|_{N^m_{\boldsymbol{\beta}}(\Omega;\mathcal{C}',\mathcal{E}')}$ for $k \geq 0$, $\emptyset \subseteq \mathcal{C}' \subseteq \mathcal{C}$ and $\emptyset \subseteq \mathcal{E}' \subseteq \mathcal{E}$.

$$\|u\|^2_{N^m_{\boldsymbol{\beta}}(\Omega;\mathcal{C}',\mathcal{E}')} := \sum_{k=0}^{m} |u|^2_{N^k_{\boldsymbol{\beta}}(\Omega;\mathcal{C}',\mathcal{E}')} \ .$$

For subdomains $K \subseteq \Omega$ denote by $|\cdot|_{N^k_{\boldsymbol{\beta}}(K;C',\mathcal{E}')}$ semi-norm with domain of integration $K \cap \Omega$.
Norm $\|\cdot\|_{N^m_{\boldsymbol{\beta}}(K;\mathcal{C}',\mathcal{E}')}$. For $m \geq 2$ holds

$$H^1_0(\Omega) \subset M^m_{\boldsymbol{\beta}}(\Omega) := N^m_{\boldsymbol{\beta}}(\Omega;\mathcal{C},\mathcal{E}) \subset N^m_{\boldsymbol{\beta}}(\Omega;\mathcal{C}',\mathcal{E}') \subset N^m_{\boldsymbol{\beta}}(\Omega;\emptyset,\emptyset) =: N^m_{\boldsymbol{\beta}}(\Omega) \ .$$

Notation:

$$b_{\boldsymbol{c}} := -1 - \beta_{\boldsymbol{c}}, \quad \boldsymbol{c} \in \mathcal{C} \ , \qquad b_{\boldsymbol{e}} := -1 - \beta_{\boldsymbol{e}}, \quad \boldsymbol{e} \in \mathcal{E} \ .$$

Analytic Regularity of Elliptic Problems in Polyhedra

Weighted Sobolev Spaces IV:

Countably Normed Spaces in Polyhedra
For $\emptyset \subseteq \mathcal{C}' \subseteq \mathcal{C}$ and $\emptyset \subseteq \mathcal{E}' \subseteq \mathcal{E}$ and $\Omega' \subseteq \Omega$,

$$B_{\boldsymbol{\beta}}(\Omega';\mathcal{C}',\mathcal{E}') := \left\{ v \in \bigcap_{m>k_{\boldsymbol{\beta}}} N^m_{\boldsymbol{\beta}}(\Omega';\mathcal{C}',\mathcal{E}') \mid |v|_{N^k_{\boldsymbol{\beta}}(\Omega';\mathcal{C}',\mathcal{E}')} \leq C_v^{k+1}\Gamma(k+1) \quad \forall k > k_{\boldsymbol{\beta}} \right\} ,$$

with

$$k_{\boldsymbol{\beta}} := -\min\{\min_{\boldsymbol{c}\in\mathcal{C}} \beta_{\boldsymbol{c}}, \min_{\boldsymbol{e}\in\mathcal{E}} \beta_{\boldsymbol{e}}\} \ .$$

Remarks:

- Spaces introduced by M. Costabel and M. Dauge in (2009)
- Definition in terms of edge-adapted, cartesian coordinates $\boldsymbol{x}^{\parallel}$ and $\boldsymbol{x}^{\perp}$.
- For $-2 < \beta_{\boldsymbol{c}}, \beta_{\boldsymbol{e}} < -1$ and for $\mathcal{C} = \mathcal{E} = \emptyset$, $B_{\boldsymbol{\beta}}(\Omega;\emptyset,\emptyset)$
  equivalent to the spaces introduced by Babuška and Guo in (1996) [Proc Roy Soc Edinburgh 1997]
- Elliptic analytic regularity shift by M. Costabel and M. Dauge and S. Nicaise [M3AS 2012]

Analytic Regularity of Elliptic Problems in Polyhedra

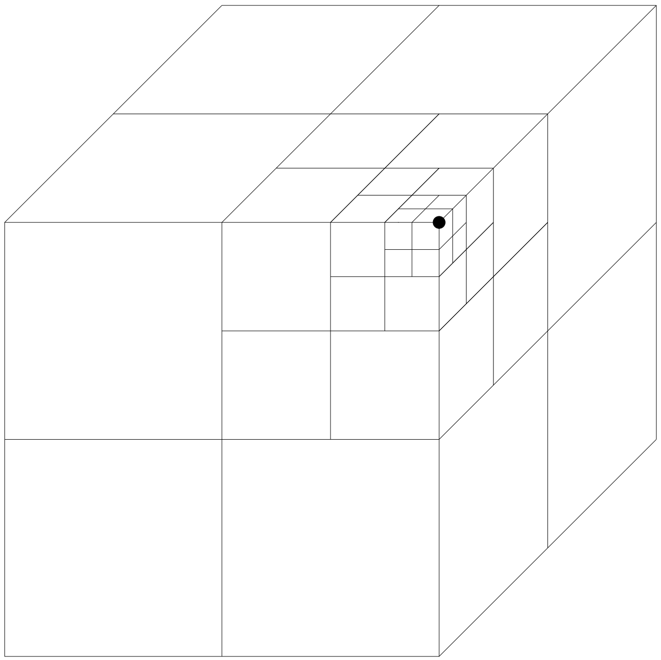


Figure 1: Three basic geometric mesh subdivisions in reference patch $\widetilde{Q}$ with subdivision ratio $\sigma = \frac{1}{2}$:
isotropic towards corner $\boldsymbol{c}$ (left),
anisotropic towards edge $\boldsymbol{e}$ (center), and
anisotropic towards one corner-edge pair $\boldsymbol{ce}$ (right).

Analytic Regularity of Elliptic Problems in Polyhedra

Analytic Regularity Shift in Polyhedron Ω:

Theorem (Analytic Regularity Shift) [Costabel, Dauge, Nicaise (2012), Babuška, Guo (1997)]
Let $\emptyset \subseteq \mathcal{E}_D \subseteq \mathcal{E}$.

Then there are $b_{\mathcal{E}}, b_{\mathcal{C}} > 0$ (depending on Ω, the coefficient matrix $\boldsymbol{A}$ and the set $\mathcal{E}_D$) such that for weight exponent vectors $\boldsymbol{b} = -1 - \boldsymbol{\beta}$ satisfying $0 < b_{\boldsymbol{c}} < b_{\mathcal{C}}$, $0 < b_{\boldsymbol{e}} < b_{\mathcal{E}}$, $\boldsymbol{c} \in \mathcal{C}$, $\boldsymbol{e} \in \mathcal{E}$,

$$f \in B_{1-\boldsymbol{b}}(\Omega; \mathcal{C}, \mathcal{E}_D) \implies u \in B_{-1-\boldsymbol{b}}(\Omega; \mathcal{C}, \mathcal{E}_D) \; .$$

hp DG FEM: geometric meshes

Geometric Mesh Patches I [Schötzau, CS and Wihler SINUM(2013)]:

- Partition Ω into $\mathfrak{P}$ open [axiparallel] hexahedral *patches* $\{Q_\mathfrak{p}\}_{\mathfrak{p}=1}^{\mathfrak{P}}$ which constitute the patch mesh $\mathcal{M}^0$,
- each $Q_\mathfrak{p} \in \mathcal{M}^0$ is an affine [orthogonal] image $Q_\mathfrak{p} = G_\mathfrak{p}(\widetilde{Q})$ of the *reference patch* $\widetilde{Q} = (-1,1)^3$,
- $\mathcal{M}^0$ regular, simplicial mesh in Ω,
- (closure of) each patch intersects either with at most one corner $\boldsymbol{c} \in \mathcal{C}$, and with either none, one or several edges $\boldsymbol{e} \in \mathcal{E}_{\boldsymbol{c}}$ meeting in $\boldsymbol{c}$,
- boundary faces on the patch $Q_\mathfrak{p}$ belong to exactly one boundary plane $\Gamma_{\boldsymbol{\iota}}$.

hp DG FEM: geometric meshes

Geometric Mesh Patches II:

- With each $Q_{\mathfrak{p}} \in \mathcal{M}^0$, associate a *geometric reference mesh patch* $\widetilde{\mathcal{M}}_{\mathfrak{p}}$ *on* $\widetilde{Q}$.
- Geometric patch meshes generated recursively by iterating *four basic* hp*-extensions* (Ex1)–(Ex4) on initial mesh $\mathcal{M}^0$: *four geometric mesh patch types* $\mathfrak{t} \in \{\boldsymbol{c}, \boldsymbol{e}, \boldsymbol{ce}, \mathrm{int}\}$ on $\widetilde{Q}$

$$\widetilde{\mathcal{M}}_{\mathfrak{p}} \in \widetilde{\mathcal{RP}} := \{\widetilde{\mathcal{M}}^{\ell,\boldsymbol{c}}_{\sigma}, \widetilde{\mathcal{M}}^{\ell,\boldsymbol{e}}_{\sigma}, \widetilde{\mathcal{M}}^{\ell,\boldsymbol{ce}}_{\sigma}, \widetilde{\mathcal{M}}^{\ell,\mathrm{int}}_{\sigma}\} = \{\widetilde{\mathcal{M}}^{\ell,\mathfrak{t}}_{\sigma}\}_{\mathfrak{t}\in\{\boldsymbol{c},\boldsymbol{e},\boldsymbol{ce},\mathrm{int}\}} \,.$$

- Whenever $Q_{\mathfrak{p}}$ abuts at $\mathcal{S}$, assign to $\widetilde{\mathcal{M}}_{\mathfrak{p}}$ (a suitably rotated and oriented) geometrically refined reference mesh patch shown in Figure 1 and denoted by
  - $\widetilde{\mathcal{M}}^{\ell,\boldsymbol{c}}_{\sigma}$ (corner patch),
  - $\widetilde{\mathcal{M}}^{\ell,\boldsymbol{e}}_{\sigma}$ (edge patch),
  - $\widetilde{\mathcal{M}}^{\ell,\boldsymbol{ce}}_{\sigma}$ (corner-edge patch), respectively.

hp DG FEM: geometric meshes

Geometric Mesh Patches III: Geometric reference patch meshes $\widetilde{\mathcal{M}}_\sigma^{\ell,\mathfrak{t}} \in \widetilde{\mathcal{RP}}$, $\mathfrak{t} \in \{\boldsymbol{c}, \boldsymbol{e}, \boldsymbol{ce}\}$

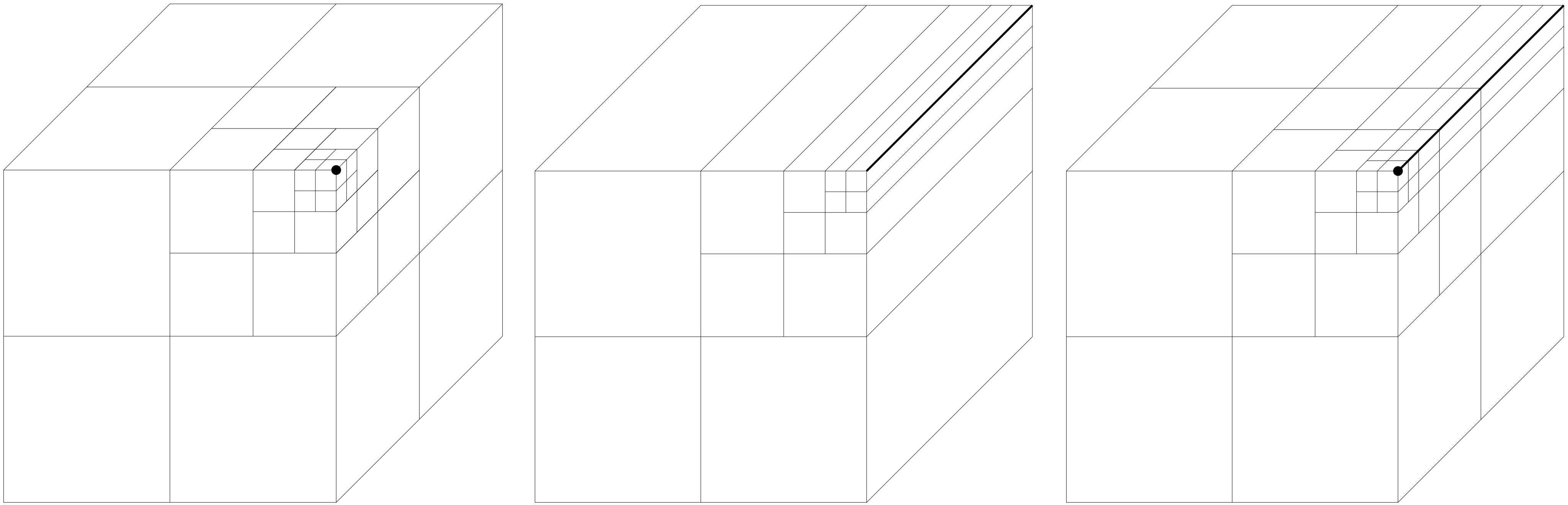


Figure 1: Three basic geometric mesh subdivisions in reference patch $\widetilde{Q}$ with subdivision ratio $\sigma = \frac{1}{2}$:
isotropic towards corner $\boldsymbol{c}$ (left),
anisotropic towards edge $\boldsymbol{e}$ (center), and
anisotropic towards one corner-edge pair $\boldsymbol{ce}$ (right). $\boldsymbol{c}, \boldsymbol{e}, \boldsymbol{ce}$ shown in boldface.

hp DG FEM: geometric meshes

Geometric Mesh Patches IV:

- σ-geometric mesh on Ω: For $\sigma \in (0,1)$ and $\ell \in \mathbb{N}$, define **σ-geometric mesh family**

$$\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell \geq 1}\ , \qquad \mathcal{M}_\sigma^{(\ell)} := \dot{\bigcup}_{\mathfrak{p}=1}^{\mathfrak{P}} \mathcal{M}_{\mathfrak{p}}\ , \qquad \mathcal{M}_{\mathfrak{p}} = G_{\mathfrak{p}}(\widetilde{\mathcal{M}}_{\mathfrak{p}})\ , \quad \widetilde{\mathcal{M}}_{\mathfrak{p}} \in \widetilde{\mathcal{RP}}\ . \tag{2}$$

- For $\mathcal{M} \in \mathfrak{M}_\sigma$, each $K \in \mathcal{M}$ is image of $\widehat{K} := (-1,1)^3$ [reference cube], under an element mapping $\Phi_K : \widehat{K} \to K$; *the composition of the corresponding patch map $G_{\mathfrak{p}}$ with an anisotropic dilation-translation*.

- Any σ-geometric mesh family $\mathfrak{M}_\sigma$ obtained by iterating the basic hp-extensions (Ex1)–(Ex4) can be partitioned into a countable sequence of disjoint mesh layers $\{\mathfrak{L}_\sigma^j\}_{j=0}^{\ell-1}$, and a corresponding nested sequence of terminal layers $\mathfrak{T}_\sigma^\ell$, such that each $\mathcal{M}_\sigma^{(\ell)} \in \mathfrak{M}_\sigma$, $\ell \geq 1$, can be written as

$$\mathcal{M}_\sigma^{(\ell)} = \mathfrak{L}_\sigma^0 \,\dot\cup\, \mathfrak{L}_\sigma^1 \,\dot\cup \ldots \dot\cup\, \mathfrak{L}_\sigma^{\ell-1} \,\dot\cup\, \mathfrak{T}_\sigma^\ell. \tag{3}$$

  Elements in the submesh

$$\mathfrak{O}_\sigma^\ell := \mathfrak{L}_\sigma^0 \,\dot\cup\, \mathfrak{L}_\sigma^1 \,\dot\cup \ldots \dot\cup\, \mathfrak{L}_\sigma^{\ell-1} \subset \mathcal{M}_\sigma^{(\ell)} \in \mathfrak{M}_\sigma, \qquad \ell \geq 1, \tag{4}$$

  are bounded away from $\mathcal{C} \cup \mathcal{E}$, while elements in terminal layer $\mathfrak{T}_\sigma^\ell$ have nontrivial intersection with $\mathcal{C} \cup \mathcal{E}$.

hp DG FEM: geometric meshes

Geometric Mesh Patches V:

- Partition $\mathcal{M}_\sigma^{(\ell)} \in \mathfrak{M}_\sigma$ into interior elements $\mathfrak{O}_\sigma^\ell$ and terminal layer elements $\mathfrak{T}_\sigma^\ell$ at $\mathcal{S}$:

$$\mathcal{M}_\sigma^{(\ell)} := \mathfrak{O}_\sigma^\ell \,\dot{\cup}\, \mathfrak{T}_\sigma^\ell,$$

  with $\mathfrak{O}_\sigma^\ell := \{ K \in \mathcal{M}_\sigma^{(\ell)} : \overline{K} \cap \mathcal{S} = \emptyset \}$ and $\mathfrak{T}_\sigma^\ell := \{ K \in \mathcal{M}_\sigma^{(\ell)} : \overline{K} \cap \mathcal{S} \neq \emptyset \}$.

- partition $\mathfrak{O}_\sigma^\ell$ into **discrete corner, edge and corner-edge neighborhoods** as $\mathfrak{O}_\sigma^\ell = \mathfrak{O}_\mathcal{C}^\ell \,\dot{\cup}\, \mathfrak{O}_\mathcal{E}^\ell \,\dot{\cup}\, \mathfrak{O}_{\mathcal{C}\mathcal{E}}^\ell \,\dot{\cup}\, \mathfrak{O}_{\mathrm{int}}^\ell$:

$$\begin{aligned}
\mathfrak{O}_{\mathrm{int}}^\ell &:= \left\{ K \in \mathfrak{O}_\sigma^\ell \,:\, \overline{K} \cap \Omega_0 \neq \emptyset \right\}, \\
\mathfrak{O}_\mathcal{C}^\ell &:= \left\{ K \in \mathfrak{O}_\sigma^\ell \,:\, \overline{K} \cap \Omega_\mathcal{C} \neq \emptyset \right\} \setminus \mathfrak{O}_{\mathrm{int}}^\ell, \\
\mathfrak{O}_\mathcal{E}^\ell &:= \left\{ K \in \mathfrak{O}_\sigma^\ell \,:\, \overline{K} \cap \Omega_\mathcal{E} \neq \emptyset \right\} \setminus (\mathfrak{O}_{\mathrm{int}}^\ell \cup \mathfrak{O}_\mathcal{C}^\ell), \\
\mathfrak{O}_{\mathcal{C}\mathcal{E}}^\ell &:= \left\{ K \in \mathfrak{O}_\sigma^\ell \,:\, \overline{K} \cap \Omega_{\mathcal{C}\mathcal{E}} \neq \emptyset \right\} \setminus \left(\mathfrak{O}_{\mathrm{int}}^\ell \cup \mathfrak{O}_\mathcal{C}^\ell \cup \mathfrak{O}_\mathcal{E}^\ell\right).
\end{aligned}$$

- Partition $\mathfrak{T}_\sigma^\ell$ into $\mathfrak{T}_\sigma^\ell := \mathfrak{T}_\mathcal{C}^\ell \,\dot{\cup}\, \mathfrak{T}_\mathcal{E}^\ell$, where

$$\mathfrak{T}_\mathcal{C}^\ell := \bigcup_{\boldsymbol{c} \in \mathcal{C}} \mathfrak{T}_{\boldsymbol{c}}^\ell, \quad \mathfrak{T}_{\boldsymbol{c}}^\ell := \{ K \in \mathfrak{T}_\sigma^\ell \,:\, \overline{K} \cap \boldsymbol{c} \neq \emptyset \}, \tag{5}$$

$$\mathfrak{T}_\mathcal{E}^\ell := \bigcup_{\boldsymbol{e} \in \mathcal{E}} \mathfrak{T}_{\boldsymbol{e}}^\ell, \quad \mathfrak{T}_{\boldsymbol{e}}^\ell := \{ K \in \mathfrak{T}_\sigma^\ell \setminus \mathfrak{T}_\mathcal{C}^\ell \,:\, (\overline{K} \cap \boldsymbol{e})^\circ \text{ is an entire edge of } K \}. \tag{6}$$

hp DG FEM: Subspaces

hp**-Finite Element Spaces I**:

- To $K \in \mathcal{M}^\ell_\sigma$ assign *anisotropic polynomial degree vector* $\boldsymbol{p}_K := (p^\perp_K, p^\parallel_K)$, with degrees $1 \le p^\perp_K \le p^\parallel_K$.
- For $K \in \mathcal{M}^\ell_\sigma$, define elemental tensor-product polynomial space

$$\mathbb{Q}_{\boldsymbol{p}_K}(K) := \{\, v \in L^2(K) \,:\, v|_K \circ \Phi_K \in \mathbb{Q}_{\boldsymbol{p}_K}(\widehat{K}) \,\}, \tag{7}$$

$\mathbb{Q}_{\boldsymbol{p}_K}(\widehat{K})$ **anisotropic elemental tensor-product polynomial space** on $\widehat{K} = \widehat{I}^3$ with $\widehat{I} = (-1, 1)$:

$$\mathbb{Q}_{\boldsymbol{p}_K}(\widehat{K}) := \mathbb{Q}_{p^\perp_K}(\widehat{I}^2) \otimes \mathbb{P}_{p^\perp_K}(\widehat{I}) = \mathbb{P}_{p^\perp_K}(\widehat{I}) \otimes \mathbb{P}_{p^\perp_K}(\widehat{I}) \otimes \mathbb{P}_{p^\parallel_K}(\widehat{I}). \tag{8}$$

- Polynomial degree vector $\boldsymbol{p}_K$ *isotropic* if $p^\perp_K = p^\parallel_K = p_K$.
- Polynomial degree distribution: $\boldsymbol{p} := \{\, \boldsymbol{p}_K : K \in \mathcal{M} \,\}$ on $\mathcal{M} \in \mathfrak{M}_\sigma$.
- Set $|\boldsymbol{p}| := \max_{K \in \mathcal{M}} |\boldsymbol{p}_K|$, with $|\boldsymbol{p}_K| := \max\{p^\perp_K, p^\parallel_K\}$.
- hp-extensions (Ex1)–(Ex4) provide $\mathfrak{s}$*-linear polynomial degree distributions* $\boldsymbol{p}_{\mathfrak{s}}(\widetilde{\mathcal{M}}^{\ell,\mathfrak{t}}_\sigma)$ on geometric reference patch meshes $\widetilde{\mathcal{M}}^{\ell,\mathfrak{t}}_\sigma$ for $\mathfrak{t} \in \{\boldsymbol{c}, \boldsymbol{e}, \boldsymbol{ce}, \mathrm{int}\}$, which increase $\mathfrak{s}$-linearly and possibly anisotropically away from singularities for a *slope parameter* $\mathfrak{s} > 0$

hp DG FEM: Subspaces

hp-Finite Element Spaces II:

- Generic discontinuous space

$$V^0(\mathcal{M},\boldsymbol{p}) := \left\{\, v \in L^2(\Omega) \,:\, v|_K \in \mathbb{Q}_{\boldsymbol{p}_K}(K),\ K \in \mathcal{M} \,\right\}.$$

- let $\mathfrak{M}_\sigma$ be a σ-geometric mesh family on Ω for some $0<\sigma<1$,
- for $\mathfrak{s}>0$, let $\{\boldsymbol{p}_{\mathfrak{s}}(\mathcal{M}^\ell_\sigma)\}_{\ell\geq 1}$ denote corresponding $\mathfrak{s}$-linear polynomial degree distributions.
- Non-conforming hp-FE spaces:

$$V^{\ell,0}_{\sigma,\mathfrak{s}} := V^0(\mathcal{M}^\ell_\sigma, \boldsymbol{p}_{\mathfrak{s}}(\mathcal{M}^\ell_\sigma)) \;, \qquad \mathcal{M}^\ell_\sigma \in \mathfrak{M}_\sigma \;.$$

- Conforming hp-FE spaces:

$$V^{\ell,1}_{\sigma,\mathfrak{s}} := V^1(\mathcal{M}^\ell_\sigma, \boldsymbol{p}_{\mathfrak{s}}(\mathcal{M}^\ell_\sigma)) = V^{\ell,0}_{\sigma,\mathfrak{s}} \cap H^1(\Omega) \qquad \mathcal{M}^\ell_\sigma \in \mathfrak{M}_\sigma \;,$$

- Constant polynomial degrees: for $k=0,1$,

$$V^{\ell,k}_{\sigma,\mathfrak{s}} := V^k(\mathcal{M}^\ell_\sigma, p) \;, \qquad hp\text{-FEM: } p \sim \ell \;.$$

hp DG FEM: Subspaces

hp-Finite Element Spaces III:
Interpatch compatibility: [for $V^{\ell,1}_{\sigma,\mathfrak{s}}$]
For $\mathfrak{p} \neq \mathfrak{p}'$, let $Q_{\mathfrak{p}}, Q_{\mathfrak{p}'} \in \mathcal{M}^0$ be two distinct patches with non-empty intersection

$$\Gamma_{\mathfrak{p}\mathfrak{p}'} := \overline{Q}_{\mathfrak{p}} \cap \overline{Q}_{\mathfrak{p}'} \neq \emptyset \ .$$

Parametrizations induced by the patch maps on the patch interfaces $\Gamma_{\mathfrak{p}\mathfrak{p}'}$ coincide "from either side":

$$G_{\mathfrak{p}} \circ \left(G^{-1}_{\mathfrak{p}'} |_{\Gamma_{\mathfrak{p}\mathfrak{p}'}} \right) = G_{\mathfrak{p}'} \circ \left(G^{-1}_{\mathfrak{p}} |_{\Gamma_{\mathfrak{p}\mathfrak{p}'}} \right) \ , \quad \mathfrak{p}, \mathfrak{p}' \in \mathfrak{P} \ , \quad \mathfrak{p} \neq \mathfrak{p}' \ .$$

$\mathcal{M}_{\mathfrak{p}}, \mathcal{M}_{\mathfrak{p}'}$ are assumed to coincide on $\Gamma_{\mathfrak{p}\mathfrak{p}'}$.

hp DG FEM: Formulation

Face Operators:

For $\mathcal{M} \in \mathfrak{M}_\sigma$, consider interior face $F = \left(\partial K^\sharp \cap \partial K^\flat\right)^\circ \in \mathcal{F}_I(\mathcal{M})$ shared by two elements $K^\sharp, K^\flat \in \mathcal{M}$.

Jumps and averages of v and $\boldsymbol{w}$ along F: with $\boldsymbol{n}_K$ outward unit normal vector on ∂K, $K \in \mathcal{M}$

$$
\begin{aligned}
[\![v]\!] &= v|_{K^\sharp}\boldsymbol{n}_{K^\sharp} + v|_{K^\flat}\boldsymbol{n}_{K^\flat} & \langle\!\langle v\rangle\!\rangle &= {}^1\!/\!_2\left(v|_{K^\sharp} + v|_{K^\flat}\right) \\
[\![\boldsymbol{w}]\!] &= \boldsymbol{w}|_{K^\sharp}\cdot\boldsymbol{n}_{K^\sharp} + \boldsymbol{w}|_{K^\flat}\cdot\boldsymbol{n}_{K^\flat} & \langle\!\langle \boldsymbol{w}\rangle\!\rangle &= {}^1\!/\!_2\left(\boldsymbol{w}|_{K^\sharp} + \boldsymbol{w}|_{K^\flat}\right).
\end{aligned}
$$

Boundary face $F = (\partial K \cap \partial\Omega)^\circ \in \mathcal{F}_B(\mathcal{M})$ for $K \in \mathcal{M}$, and sufficiently smooth functions $v, \boldsymbol{w}$ on K,

$$
[\![v]\!] = v|_K\boldsymbol{n}_\Omega, [\![\boldsymbol{w}]\!] = \boldsymbol{w}|_K\cdot\boldsymbol{n}_\Omega, \quad \text{and} \quad \langle\!\langle v\rangle\!\rangle = v|_K, \langle\!\langle \boldsymbol{w}\rangle\!\rangle = \boldsymbol{w}|_K,
$$

$\boldsymbol{n}_\Omega$ outward unit normal vector on $\partial\Omega$.

hp DG FEM: Formulation

hp-IP DG Discretization:

$$u_{\mathrm{DG}} \in V^{\ell,0}_{\sigma,\mathfrak{s}} : \qquad a_{\mathrm{DG}}(u_{\mathrm{DG}}, v) = \int_\Omega f v \,\mathrm{d}\boldsymbol{x} \qquad \forall\, v \in V^{\ell,0}_{\sigma,\mathfrak{s}},$$

where $a_{\mathrm{DG}}(u,v)$ is given by

$$\begin{aligned} a_{\mathrm{DG}}(w,v) = &\int_\Omega (\boldsymbol{A}\nabla_h w)\cdot\nabla_h v\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{F}(\mathcal{M})} \langle\!\langle \boldsymbol{A}\nabla_h w\rangle\!\rangle \cdot [\![v]\!]\,\mathrm{d}s \\ &+ \theta \int_{\mathcal{F}(\mathcal{M})} \langle\!\langle \boldsymbol{A}\nabla_h v\rangle\!\rangle \cdot [\![w]\!]\,\mathrm{d}s + \gamma \int_{\mathcal{F}(\mathcal{M})} \alpha [\![v]\!]\cdot[\![w]\!]\,\mathrm{d}s. \end{aligned}$$

$\alpha \in L^\infty(\mathcal{F})$ **discontinuity stabilization function**:

$$\alpha_F = \begin{cases} \dfrac{\max\left(p^\perp_{K_\sharp,F}, p^\perp_{K_\flat,F}\right)^2}{\min\left(h^\perp_{K_\sharp,F}, h^\perp_{K_\flat,F}\right)} & \text{if } F = (\partial K_\sharp \cap \partial K_\flat)^\circ \in \mathcal{F}_I(\mathcal{M}), \\ \dfrac{(p^\perp_{K,F})^2}{h^\perp_{K,F}} & \text{if } F = (\partial K \cap \partial\Omega)^\circ \in \mathcal{F}_B(\mathcal{M}) \,. \end{cases}$$

$\theta = -1$: SIP, $\theta = 1$ NIP.

hp DG FEM: Formulation

hp-IP DG Discretization: Stability and Galerkin Orthogonality
DG norm:

$$\forall v \in V^{\ell,0}_{\sigma,\mathfrak{s}} + H^1(\Omega): \quad \|v\|^2_{\mathrm{DG}} = \int_\Omega |\nabla_h v|^2 \,\mathrm{d}\boldsymbol{x} + \gamma \int_{\mathcal{F}} \alpha \, |[\![v]\!]|^2 \,\mathrm{d}s \,.$$

Theorem: For any $\mathcal{M} \in \mathfrak{M}_\sigma$ with $0 < \sigma < 1$ and $\mathfrak{s}$-linear any degree vector $\boldsymbol{p}$, $a_{\mathrm{DG}}(\cdot,\cdot)$ is continuous and coercive on $V^{\ell,0}_{\sigma,\mathfrak{s}}$: exist $0 < C_1 \leq C_2 < \infty$ independent of ℓ, the element aspect ratios, the local mesh sizes, and the local polynomial degree vectors such that

$$|a_{\mathrm{DG}}(v,w)| \leq C_1 \|v\|_{\mathrm{DG}} \|w\|_{\mathrm{DG}} \qquad \forall\, v,w \in V^{\ell,0}_{\sigma,\mathfrak{s}} \,.$$

For $\gamma > 0$ sufficiently large (ind. of ℓ, element aspect ratios, local mesh sizes, local polynomial degree vectors),

$$a_{\mathrm{DG}}(v,v) \geq C_2 \|v\|^2_{\mathrm{DG}} \qquad \forall\, v \in V^{\ell,0}_{\sigma,\mathfrak{s}} \,.$$

In particular, there exists a unique DG solution $u_{DG} \in V^{\ell,0}_{\sigma,\mathfrak{s}}$.
Galerkin orthogonality: for $u \in N^2_{-1-\boldsymbol{\beta}}(\Omega;\mathcal{C},\mathcal{E}_D)$ where $\boldsymbol{\beta}$ is a suitable weight vector,
the DG approximation $u_{\mathrm{DG}} \in V^{\ell,0}_{\sigma,\mathfrak{s}}$ satisfies

$$a_{\mathrm{DG}}(u - u_{\mathrm{DG}}, v) = 0 \qquad \forall\, v \in V^{\ell,0}_{\sigma,\mathfrak{s}} \,.$$

hp DG FEM

Anisotropic trace inequality [Schötzau, CS and Wihler, SINUM2013]:
Let $\mathcal{M} \in \mathfrak{M}_\sigma$ for $0 < \sigma < 1$, $K \in \mathcal{M}$ axiparallel, $F \in \mathcal{F}_K$ and $s \geq 1$. Then

$$\forall \in W^{1,s}(K) : \quad \|v\|^s_{L^s(F)} \leq C_s \left(h^\perp_{K,F}\right)^{-1} \left(\|v\|^s_{L^s(K)} + \left(h^\perp_{K,F}\right)^s \|\partial_{K,F,\perp} v\|^s_{L^s(K)}\right) .$$

$C_s > 0$ depends only on σ, but is independent of the element size and element aspect ratio.

$\partial_{K,f,\perp}$ partial derivative in direction transversal to $F \in \mathcal{F}_K$.

In 2d used by [Georgoulis, Hall, Houston (2007)].

hp DG FEM

DG 'Quasioptimality' I (Dirichlet Problem) [Schötzau, CS and Wihler, SINUM2013]:

Assume $u \in M^2_{-1-\boldsymbol{\beta}}(\Omega)$. Then, on any $\mathcal{M}^\ell_\sigma \in \mathfrak{M}_\sigma$, and for any family of polynomial degree vectors $\{\boldsymbol{p}(\mathcal{M}^{(\ell)}_\sigma)\}_{\ell \geq 1}$ holds the error estimate

$$\|u - u_{\mathrm{DG}}\|^2_{\mathrm{DG}} \leq C \boldsymbol{p}^4_{\max} \left(\Upsilon_{\mathfrak{O}^\ell_\sigma}[\eta] + \Upsilon_{\mathfrak{T}^\ell_\sigma}[\eta] \right),$$

with

- η interpolation error $u - \pi_b u$, with suitable hp-**base projector** π_b,
- $u_{\mathrm{DG}} \in V^{\ell,0}_{\sigma,\mathfrak{s}}$ is the DG solution,
- the consistency terms

$$\Upsilon_{\mathfrak{O}^\ell_\sigma}[\eta] = \sum_{K \in \mathfrak{O}^\ell_\sigma} \left(\max_{F \in \mathcal{F}_K} \left(h^\perp_{K,F}\right)^{-2} \|\eta\|^2_{L^2(K)} + \|\nabla \eta\|^2_{L^2(K)} \right) + \sum_{K \in \mathfrak{O}^\ell_\sigma} \sum_{F \in \mathcal{F}_K} \left(h^\perp_{K,F}\right)^2 \|\partial_{K,F,\perp} \nabla \eta\|^2_{L^2(K)},$$

$$\Upsilon_{\mathfrak{T}^\ell_\sigma}[\eta] = \sum_{K \in \mathfrak{T}^\ell_\sigma} \left(\max_{F \in \mathcal{F}_K} \left(h^\perp_{K,F}\right)^{-2} \|\eta\|^2_{L^2(K)} + \|\nabla \eta\|^2_{L^2(K)} \right) + \sum_{K \in \mathfrak{T}^\ell_\sigma} \sum_{F \in \mathcal{F}_K} |f|^{-1} h^\perp_{K,F} \|\nabla \eta\|^2_{L^1(F)} .$$

$C > 0$ ind. of refinement level ℓ, aspect ratios, local mesh sizes, and of local polynomial degree vectors.

hp DG FEM

DG 'Quasioptimality' II (Neumann Problem) [Schötzau, CS and Wihler, MathComp 2015]:
Assume $\boldsymbol{A} = \mathbf{1}$ and $u \in N^2_{-1-\boldsymbol{\beta}}(\Omega; \mathcal{C}, \mathcal{E}_D)$.

Geometric mesh $\mathcal{M}^\ell_\sigma \in \mathfrak{M}_\sigma$, polynomial degree vectors $\{\boldsymbol{p}(\mathcal{M}^{(\ell)}_\sigma)\}_{\ell \geq 1}$

$\pi_b = \Pi^\perp \otimes \Pi^\parallel$ tensorized dG interpolant on $N^2_{-1-\boldsymbol{\beta}}(\Omega; \mathcal{C}, \mathcal{E}_D)$, on $K \in \mathfrak{O}^\ell_\sigma \dot{\cup} \mathfrak{T}^\ell_{\mathcal{E}}$, and

$$\eta^\perp := u - \Pi^\perp u, \qquad \eta^\parallel := u - \Pi^\parallel u, \qquad \text{for } K \in \mathfrak{O}^\ell_\sigma \dot{\cup} \mathfrak{T}^\ell_{\mathcal{E}}.$$

For $K \in \mathfrak{O}^\ell_\sigma \dot{\cup} \mathfrak{T}^\ell_{\mathcal{E}}$, **with $\Pi^\perp$ and $\Pi^\parallel$ being L^2-projections**,

$$\eta = (u - \Pi^\parallel u) + \Pi^\parallel(u - (\Pi^\perp u)) = \eta^\parallel + \Pi^\parallel \eta^\perp \ .$$

Then

$$\|u - u_{\rm DG}\|^2_{\rm DG} \leq C\boldsymbol{p}^{12}_{\max} \Bigg(\Upsilon_{\mathfrak{O}^\ell_\sigma}[\eta^\perp] + \Upsilon_{\mathfrak{O}^\ell_\sigma}[\eta^\parallel] + \sum_{\boldsymbol{c} \in \mathcal{C}} \Upsilon_{\mathfrak{T}^\ell_{\boldsymbol{c}}}[\eta]$$
$$+ \sum_{\boldsymbol{e} \in \mathcal{E}} \left(\Upsilon_{\mathfrak{T}^\ell_{\boldsymbol{e},1}}[\eta^\perp] + \Upsilon_{\mathfrak{T}^\ell_{\boldsymbol{e},1}}[\eta^\parallel] + \Upsilon_{\mathfrak{T}^\ell_{\boldsymbol{e},2}}[\eta] \right) + \sum_{\boldsymbol{e} \in \mathcal{E}_D} \left(\Upsilon_{\mathfrak{T}^\ell_{\boldsymbol{e},D}}[\eta^\perp] + \Upsilon_{\mathfrak{T}^\ell_{\boldsymbol{e},D}}[\eta^\parallel] \right) \Bigg).$$

$C > 0$ independent of the refinement level ℓ, the local mesh sizes and the local polynomial degree vectors.

hp DG FEM

DG 'Quasioptimality' III (Exponential Convergence) [Schötzau, CS and Wihler, MathComp 2015]:

Assume

- $u \in B^2_{-1-\boldsymbol{b}}(\Omega; \mathcal{C}, \mathcal{E}_D) \cap H^{1+\theta}(\Omega)$, some $\theta > 0$,
- Geometric mesh $\mathcal{M}^\ell_\sigma \in \mathfrak{M}_\sigma$,
- linear polynomial degree vectors $\{\boldsymbol{p}(\mathcal{M}^{(\ell)}_\sigma)\}_{\ell \geq 1}$,
- $\pi_b = \Pi^\perp \otimes \Pi^\parallel$ tensorized dG interpolant on $N^2_{-1-\boldsymbol{\beta}}(\Omega; \mathcal{C}, \mathcal{E}_D)$,

Then for $\ell \geq 1$, the hp-dG approximation $u_{\mathrm{DG}} \in V^{\ell,0}_{\sigma,\mathfrak{s}}$ is well-defined.

$$\|\!|u - u_{DG}|\!\|_{\mathrm{DG}} \leq C \exp\left(-b\sqrt[5]{N}\right), \qquad N = \dim(V^{\ell,0}_{\sigma,\mathfrak{s}}) \; .$$

The constants $b > 0$ and $C > 0$ are independent of N, but depend on σ, $\mathcal{M}^0$, θ, γ, $\min \boldsymbol{b} > 0$.

Remarks:

For uniform polynomial degrees, $\exp\left(-b\sqrt[5]{N}\right)$ analogous to bound of Guo (Proc. CIRM Luminy 1995), Babuška and Guo (CMAME 1996), on geom. meshes of tetrahedra.

hp DG FEM

DG 'Quasioptimality' IV (Exponential Convergence) [Schötzau, CS and Wihler, MathComp 2016]:
Elements of Proof:
1. Tensor Projector: $\mathcal{M}_\sigma^{(\ell)} = \mathfrak{O}_\sigma^\ell \dot\cup \mathfrak{T}_\mathcal{C}^\ell \dot\cup \mathfrak{T}_\mathcal{E}^\ell$:

$$(\Pi v)_K = \Pi_K v|_K := \begin{cases} \Pi_{\boldsymbol{p}_K}(v|_K) = \Pi^\perp_{p_K^\perp} \otimes \Pi^\parallel_{p_K^\parallel}(v|_K) & \text{if } K \in \mathfrak{O}_\sigma^\ell, \\ \mathcal{I}_1(v|_K) & \text{if } K \in \mathfrak{T}_\mathcal{C}^\ell, \\ \mathcal{I}_1^\perp \otimes \Pi_{p_K^\parallel}(v|_K) & \text{if } K \in \mathfrak{T}_\mathcal{E}^\ell. \end{cases} \tag{9}$$

- $\Pi_{\boldsymbol{p}_K}$ is the (scaled) L^2-projection onto $\mathbb{Q}_{\boldsymbol{p}_K}(K)$ given by

$$\Pi_{\boldsymbol{p}_K}(v|_K) := \left(\widehat{\Pi}_{\boldsymbol{p}_K}(v \circ \Phi_K)\right) \circ \Phi_K^{-1}, \tag{10}$$

  with $\widehat{\Pi}_{\boldsymbol{p}_K}$ reference projection, Φ_K affine element mapping,
- $\mathcal{I}_1$ is dG interpolant onto $\mathbb{P}_1$.

2. univariate hp-error bounds:
Lemma:

(0) For any $p, k \in \mathbb{N}$ with $p \geq 2k-1$, there is a projector $\widehat{\pi}_{p,k} : H^k(I) \to \mathbb{P}^p(I)$ that satisfies $(\widehat{\pi}_{p,k} u)^{(k)} = \widehat{\pi}_{p-k,0}(u^{(k)})$, and $(\widehat{\pi}_{p,k})^{(j)} u(\pm 1) := u^{(j)}(\pm 1)$, for any $j = 0, \ldots, k-1$.

(i) For every conformity $k \in \mathbb{N}$ exists a constant $C_k > 0$ such that

$$\forall u \in H^k(I)\ , \forall p \geq 2k-1 : \quad \|\widehat{\pi}_{p,k} u\|_{H^k(I)} \leq C_k \|u\|_{H^k(I)}\ . \tag{11}$$

(ii) For integers $p, k \in \mathbb{N}$ with $p \geq 2k-1$, $\kappa = p - k + 1$ and for $u \in H^{k+s}(I)$ with any $k \leq s \leq \kappa$ there holds the error bound

$$\|(u - \widehat{\pi}_{p,k} u)^{(j)}\|^2_{L^2(I)} \leq \frac{(\kappa - s)!}{(\kappa + s)!} \|u^{(k+s)}\|^2_{L^2(I)}, \qquad j = 0, 1, \ldots, k. \tag{12}$$

3. Anisotropic Tensor Projection:

$$\widehat{\Pi}^d_{\boldsymbol{p},\boldsymbol{k}} = \bigotimes_{i=1}^{d} \widehat{\pi}^{(i)}_{p_i,k_i},$$

- $k_i = 2$ for DG base interpolant in Dirichlet case, • $k_i = 0$ for DG base interpolant in Neumann case,
- $k^{||} = 1$ and $k^{\perp} = 0$ for CG hp base interpolant.

4. Trace Commutativity: For $d \geq 2$, $k_j \geq 1$, $1 \leq j \leq d$ holds

$$\left(\widehat{\Pi}^d_{p,\boldsymbol{k}} v\right)|_{\widehat{x}_j=\pm 1} = \Big(\bigotimes_{1\leq i\neq j\leq d} \widehat{\pi}^{(i)}_{p,k_i} \Big) (v(\cdot, \widehat{x}_j = \pm 1)) \ .$$

5. Summation of interpolation error bounds over all mesh layers: [e.g. for $\mathfrak{O}_{\mathcal{C}}$]
Under the regularity assumption

$$|u|_{N^m_{-1-\boldsymbol{\beta}}(\Omega)} \leq C_u^{m+1} m! \qquad \forall\, m \in \mathbb{N}_0,$$

$$\Upsilon_{\mathfrak{O}^\ell_{\mathcal{C}}}[\eta] \lesssim \sum_{j=2}^{\ell} \sigma^{2(\ell-j)\min\boldsymbol{\beta}} \Psi_{p_j-1,s_j-1} C_u^{2s_j} \Gamma(s_j+6)^2$$

for any $s_j \in [3, p_j]$. Here,

$$\Psi_{q,r} = \frac{\Gamma(q+1-r)}{\Gamma(q+1+r)}, \qquad 0 \leq r \leq q.$$

Lemma [SSWII, SINUM 2013]:
for any $c > 0$ and any $q \in \mathbb{N}$, there exist constants $b > 0$ and $C > 0$ (depending only on c and q) such that

$$\forall\, p \geq k+1 : \qquad \min_{s\in[k+1,p]} \left\{ c^{2s} \Gamma(s+q)^2 \Psi_{p-1,s-1} \right\} \leq C^2 \exp(-2bp) \ .$$

Numerical Experiments hp DG FEM in $\mathbb{R}^3$

(Vincent Heuveline and Staffan Ronnas, IWR Heidelberg)

Software: HiFlow3d

Manufactured Singular Solutions: $\Omega = (0,1)^3$, f such that

Corner: $u(x) = r_{\boldsymbol{c}}(x)^{\alpha}$, $\alpha > -1/2$,

Edge: $u(x) = r_{\boldsymbol{e}}(x)^{\beta}$, $\beta > 0$,

CornerEdge Singularity: $u(x) = r_{\boldsymbol{c}}(x)^{\alpha} r_{\boldsymbol{e}}(x)^{\beta}$.

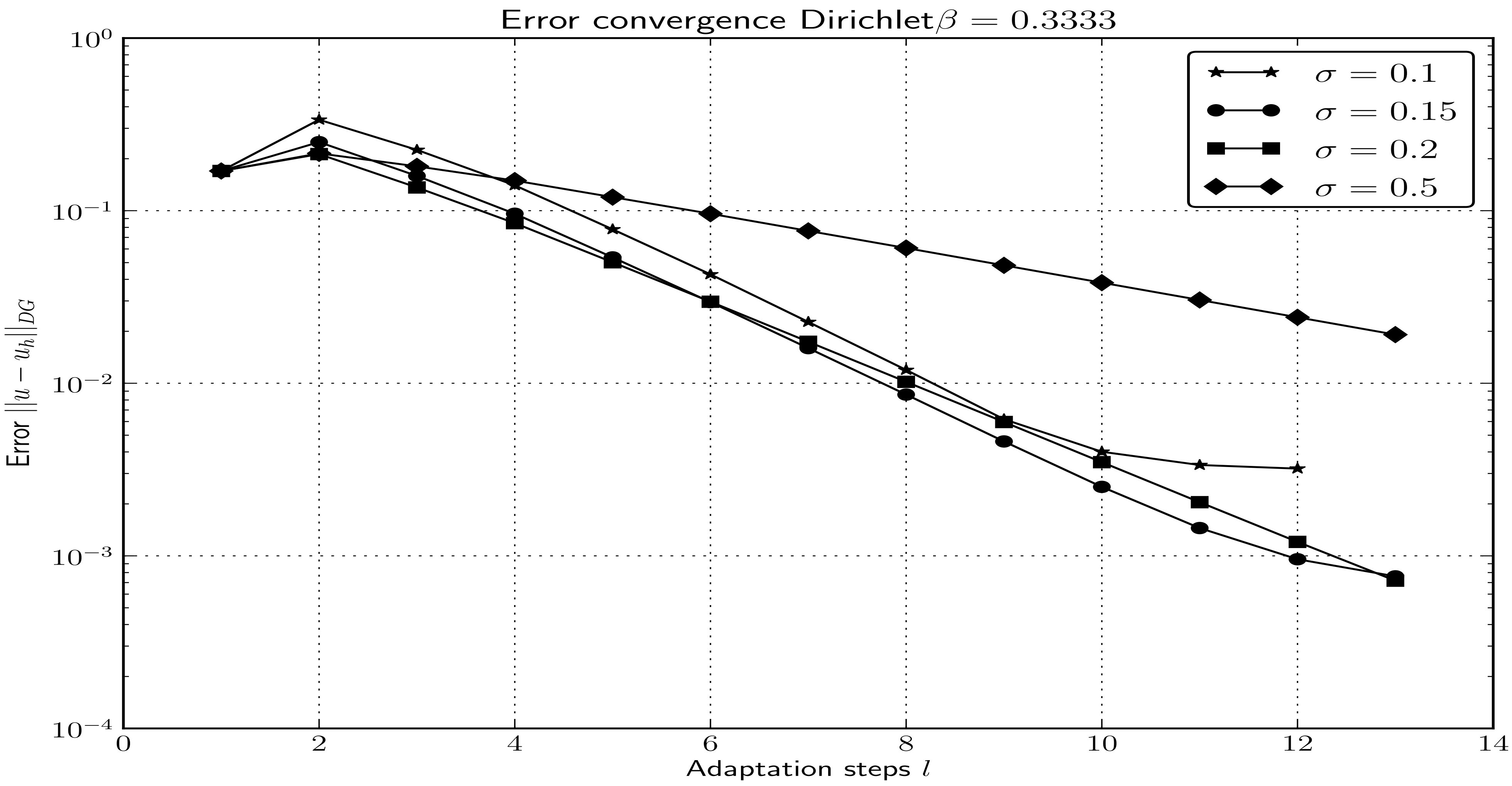


Figure 2: Exponential convergence of hp DG FEM. Edge Singularity $r_{\boldsymbol{e}}^{\beta}$ with $\beta = 0.33333$.

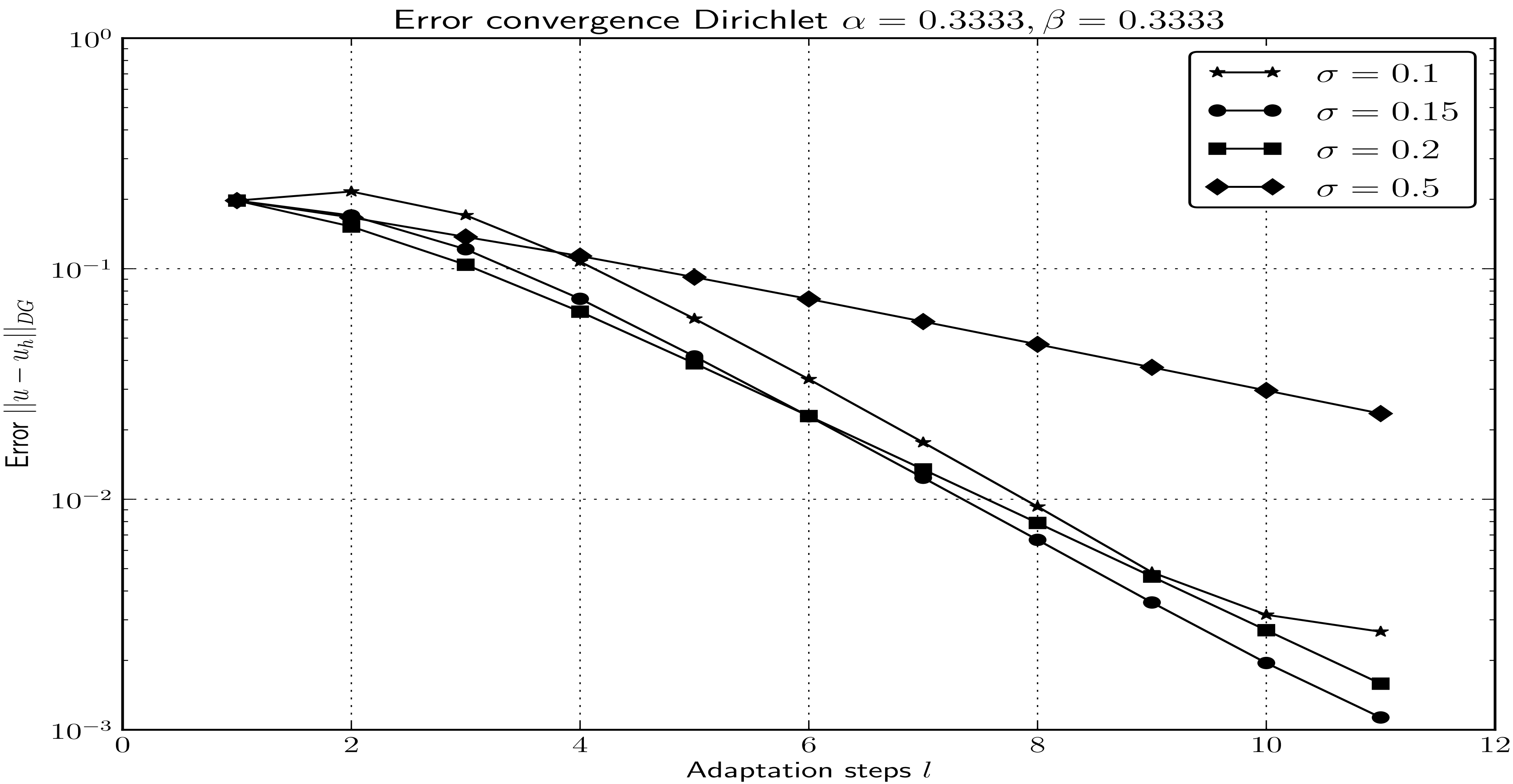


Figure 3: Exponential convergence of hp-DGFEM. Edge-Vertex Singularity $r_{\boldsymbol{c}}^{\alpha} r_{\boldsymbol{e}}^{\beta}$ with $\alpha = \beta = 0.33333$.

hp CG FEM in $\mathbb{R}^3$: Exponential Convergence

Dirichlet problem for Poisson equation in bounded, axiparallel polyhedron $\Omega \subset \mathbb{R}^3$:

$$-\nabla \cdot (\boldsymbol{A}\nabla u) = f \quad \text{in} \quad \Omega\,, \qquad u|_{\partial\Omega} = 0\,.$$

hp CG FEM:

$$u^\ell \in V^{\ell,1}_{\sigma,\mathfrak{s}}: \quad (\nabla v, \boldsymbol{A}\nabla u^\ell) = (f,v) \quad \forall v \in V^{\ell,1}_{\sigma,\mathfrak{s}}\,.$$

Theorem[Schötzau & Sc (M3AS 2015)]:
Assume Ω axiparallel and $f \in B_{1-\boldsymbol{b}}(\Omega;\mathcal{C},\mathcal{E})$. Then

$$\|u - u^\ell\|_{H^1(\Omega)} \leq C \exp(-bN_\ell^{1/5})\,.$$

Remarks:

- numerically observed by B. Andersson at FFA Sweden, 3d hp code STRIPE (1990-),
- $b > 0$ strongly depends on σ and on slope of polynomial degree vectors.
- $b(\sigma;\alpha) > 0$ only numerically, Num. Experiments: $0 < \sigma < 1/2$ advantageous for all singularity strengths.
- Space dimension $d = 1$, $\Omega = (0,1)$, $u(x) = x^\alpha(1-x)$: $\sigma_{opt} = (\sqrt{2}-1)^2 \sim 0.17...$, $\mathfrak{s}_{opt} = 0.39...(\alpha - 1/2)$, $b(\sigma_{opt}, \mathfrak{s}_{opt}) = 1.5632..\sqrt{\alpha - 1/2}$. Gui and Babuška (1985).

hp CG FEM in $\mathbb{R}^3$: Exponential Convergence

Proof[outline]:

step 0: Quasioptimality

$$\|u - u^\ell\|_{H^1(\Omega)} \le C\|u - \pi^{\ell,1}_{\sigma,\mathfrak{s}} u\|_{H^1(\Omega)} \ ,$$

where $\pi^{\ell,1}_{\sigma,\mathfrak{s}} : H^1(\Omega) \to V^{\ell,1}_{\sigma,\mathfrak{s}}$ exponentially consistent for $u \in B_{-1-\boldsymbol{b}}(\Omega; \mathcal{C}, \mathcal{E}_D)$. Construct $\pi^{\ell,1}_{\sigma,\mathfrak{s}}$ in several steps.

step 1: dG hp**-base projector** $\pi_b : H^1(\Omega) \to V^{\ell,0}_{\sigma,\mathfrak{s}}$ and exponential convergence in broken $H^1(\Omega)$ norm, analogous to [but easier than] DG proof.

step 2: **Polynomial jump liftings**
of polynomial vertex-, edge- and face-jumps with algebraic in p stability, independent of element aspect ratio.
[L. Zhu, S. Giani, P. Houston, and D. Schötzau. M3AS **21** (2):267-306, 2011]

step 3: build hp**-reference patch projectors** $\Pi^{\ell,\mathfrak{t}}_{\sigma,\mathfrak{s}}$, for $\mathfrak{t} \in \{\boldsymbol{c}, \boldsymbol{e}, \boldsymbol{ce}\}$, well-defined on $H^1(\widetilde{Q})$, s.t.

$$\|u - \Pi^{\ell,\mathfrak{t}}_{\sigma,\mathfrak{s}}\|_{H^1(\widetilde{Q})} \le C \begin{cases} \exp(-bN_\ell^{1/4}) & \text{if} \quad \mathfrak{t} = \boldsymbol{e}, \boldsymbol{c} \ , \\ \exp(-bN_\ell^{1/5}) & \text{if} \quad \mathfrak{t} = \boldsymbol{ce} \ . \end{cases}$$

step 4: Transport to physical patch $Q_\mathfrak{p}$ by patchmap $G_\mathfrak{p}$:

$$\pi^{\ell,\mathfrak{t}}_{\sigma,\mathfrak{s}} v = \left(\Pi^{\ell,\mathfrak{t}}_{\sigma,\mathfrak{s}}(v|_{Q_\mathfrak{p}} \circ G_\mathfrak{p})\right) \circ G_\mathfrak{p}^{\ -1}$$

hp CG FEM: Construction of $\Pi^{\ell,\mathfrak{t}}_{\sigma,\mathfrak{s}}$

step 1: hp-base projector (from DG)

Proposition:

For all parameters $\sigma \in (0,1)$ and $\mathfrak{s} > 0$ there is a tensorized projector

$$\pi_b = \pi_b^{\perp} \otimes \pi_b^{\|} : H^1(\Omega) \to V^{0,\ell}_{\sigma,\mathfrak{s}},$$

which is

- well-defined on $H^1(\Omega)$,
- conforming over a set $\mathcal{F}^{\perp}_{ID}(\mathcal{M}^{\ell}_{\sigma}) \subset \mathcal{F}_{ID}(\mathcal{M}^{\ell}_{\sigma})$ of edge-perpendicular Dirichlet faces and
- generally non-conforming over a complement set $\mathcal{F}^{\|}_{ID}(\mathcal{M}^{\ell}_{\sigma}) = \mathcal{F}^{\|}_{I}(\mathcal{M}^{\ell}_{\sigma}) \cup \mathcal{F}^{\|}_{D}(\mathcal{M}^{\ell}_{\sigma})$ of edge-parallel faces F

For functions u with $u \in B_{-1-\boldsymbol{b}}(\Omega;\emptyset,\emptyset) \cap H^{1+\theta}(\Omega)$ for some $\theta \in (0,1)$ for the error terms given by

$$\eta_b := u - \pi_b u, \qquad \eta_b^{\perp} := u - \pi_b^{\perp} u, \qquad \eta_b^{\|} := u - \pi_b^{\|} u,$$

hold the exponential error bounds

$$\Upsilon^{\|}_{\mathcal{M}^{\ell}_{\sigma}}[\eta_b]^2 + \mathrm{Jmp}_{\mathcal{F}^{\|}_{ID}(\mathcal{M}^{\ell}_{\sigma})}[\eta_b]^2 + \Upsilon^{\perp}_{\mathcal{M}^{\ell,\circ}_{\sigma}}[\eta_b^{\perp}]^2 + \Upsilon^{\|}_{\mathcal{M}^{\ell,\circ}_{\sigma}}[\eta_b^{\|}]^2 \leq C \exp\big(-2b\sqrt[5]{N}\big),$$

Constants $b, C > 0$ independent of $N = \dim(V^{\ell,0}_{\sigma,\mathfrak{s}})$, but depend σ, $\mathfrak{s}$ and on θ.

Here, for a set $\mathcal{F}' \subset \mathcal{F}_{ID}(\mathcal{M})$ of faces,

$$\mathrm{Jmp}_{\mathcal{F}'}[u]^2 := \sum_{F \in \mathcal{F}'} \mathtt{h}_F^{-1} \| [\![u]\!] \|^2_{L^2(F)} \; .$$

For sets $\mathcal{M}' \subseteq \mathcal{M}^\ell_\sigma \in \mathfrak{M}_\sigma$ of axiparallel hexahedra, introduce broken H^1-norms:

$$\Upsilon^\perp_{\mathcal{M}'}[u]^2 := \sum_{K \in \mathcal{M}'} N^\perp_K[u]^2, \qquad \Upsilon^\parallel_{\mathcal{M}'}[u]^2 := \sum_{K \in \mathcal{M}'} N^\parallel_K[u]^2,$$

with **edge-scaled elemental norms** defined by

$$N^\perp_K[u]^2 := (h^\perp_K)^{-2} \|u\|^2_{L^2(K)} + \|\nabla u\|^2_{L^2(K)}, \qquad N^\parallel_K[u]^2 := (h^\parallel_K)^{-2} \|u\|^2_{L^2(K)} + \|\nabla u\|^2_{L^2(K)} \; .$$

Step 2: Polynomial vertex-, edge- and face-jump liftings

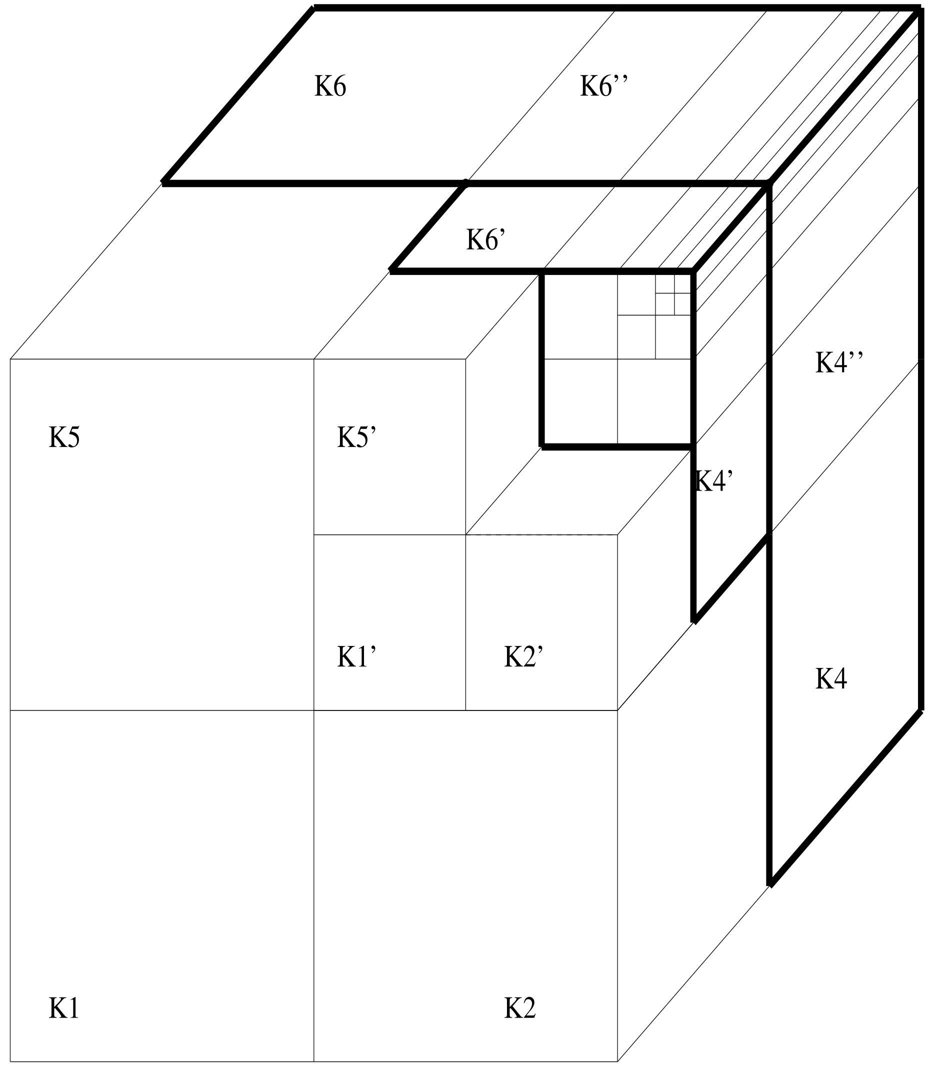


Figure 4: Patch decomposition for $\sigma = 0.5$ and $\ell = 5$.
The scaled edge-patch blocks $\widetilde{\Psi}^{\ell',\boldsymbol{ce}}(\widetilde{\mathcal{M}}_{\sigma}^{\ell',\boldsymbol{e}})$ and $\widetilde{\Psi}^{\ell'-1,\boldsymbol{ce}}(\widetilde{\mathcal{M}}_{\sigma}^{\ell'-1,\boldsymbol{e}})$ for $\sigma = 0.5$ and $\ell' = 5$.
Diagonal elements K_4, K_6 and K_4', K_6' belong to $\widetilde{\mathcal{D}}_{\sigma}^{\ell',\boldsymbol{ce}}$ and $\widetilde{\mathcal{D}}_{\sigma}^{\ell'-1,\boldsymbol{ce}}$, respectively.

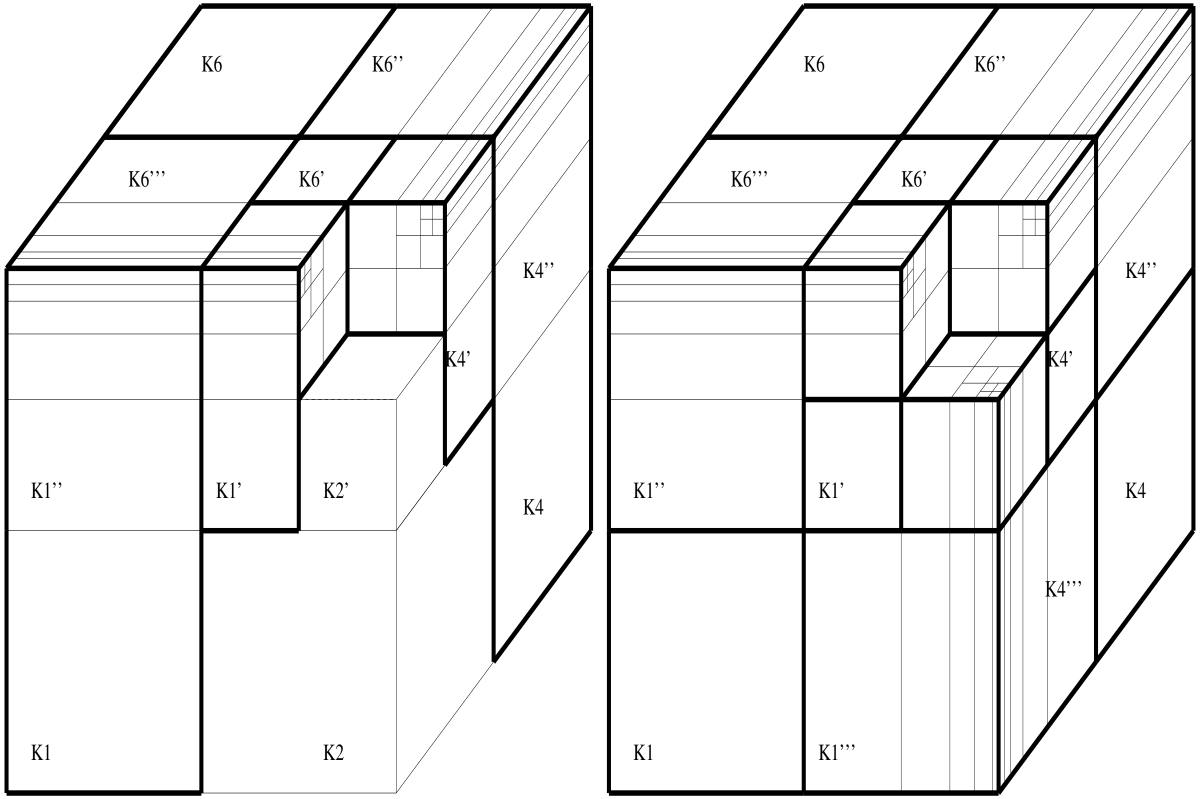


Figure 5: Scaled edge-patch blocks for $\sigma = 1/2$ and $\ell' = 5$.
Refinement along two resp. three edges with diagonal elements K_1, K_4, K_6 and K_1', K_4', K_6'.

Some corollaries...

n-width bounds, exponential convergence of MOR

Kolmogorov n-width [A.N. Kolmogorov (1941)] of compact set $\mathcal{K}$ in a Hilbert space $\mathcal{X}$:

$$d_n(\mathcal{K}, \mathcal{X}) = \inf_{\substack{V_n \subset \mathcal{X} \\ \dim(V_n) = n}} \sup_{\substack{u \in \mathcal{K} \\ \|u\|_{\mathcal{X}} \leq 1}} \inf_{v \in V_n} \|u - v\|_{\mathcal{X}}$$

Subspace V_n 'at our disposal', non-polynomial, problem adapted,...

$$d_n(\mathcal{K}, \mathcal{X}) = \sup_{\substack{u \in \mathcal{K} \\ \|u\|_{\mathcal{X}} \leq 1}} \inf_{v \in V_n^*} \|u - v\|_{\mathcal{X}}$$

[A. Pinkus (1985): n-widths in Approximation Theory, Springer Publ.]

n-width bounds, exponential convergence of MOR

- Exponentially small bounds on n-widths.

  $f \in B_{1-\boldsymbol{b}}(\Omega; \mathcal{C}, \mathcal{E}_D)$ implies solution set $\mathcal{K} = \{u : u = A^{-1}f\} \subset H^1(\Omega)$ is compact and has exponentially small Kolmogorov n-width in $H^1(\Omega)$:

$$d_n(\mathcal{K}, H^1(\Omega)) \leq C \exp(-bn^{1/5}) \ .$$

  Proof: Corollary of exponential convergence of hp-CG FEM in Ω.

$$\begin{aligned} d_n(\mathcal{K}, H^1(\Omega)) &= \inf_{V_n \subset H^1(\Omega): \dim V_n = n} \sup_{u \in \mathcal{K}} \inf_{v \in V_n} \frac{\|u - v\|_{H^1(\Omega)}}{\|u\|_{H^1(\Omega)}} \\ &\lesssim \sup_{f \in B_{1-\boldsymbol{b}}(\Omega; \mathcal{C}, \mathcal{E}_D)} \inf_{v \in V^{1,\ell}_{\sigma,\mathfrak{s}}} \frac{\|u(f) - v\|_{H^1(\Omega)}}{\|f\|_{H^{-1}(\Omega)}} \\ &\lesssim \sup_{f \in B_{1-\boldsymbol{b}}(\Omega; \mathcal{C}, \mathcal{E}_D)} \frac{\|u(f) - \pi^{\ell,1}_{\sigma,\mathfrak{s}} u(f)\|_{H^1(\Omega)}}{\|f\|_{H^{-1}(\Omega)}} \\ &\lesssim \exp(-b \dim(V^{\ell,1}_{\sigma,\mathfrak{s}})^{1/5}) \ . \end{aligned}$$

- $\Longrightarrow$ exponential convergence of greedy basis searches in RB and MOR methods
  [P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrovna SIAM J. Math. Analysis(2011)]
  [Y. Maday, O. Mula, A.T. Patera and M. Yano CMAME(2015)]

Exponential Convergence of QTT h-FEM

Theorem[Kazeev and CS 2015]:
Assume Ω straight-sided polygon in space dimension $d = 2$.

The node vector of the continuous, piecewise bilinear interpolant $I_h u$ of $u \in B_{-1-\boldsymbol{b}}(\Omega)$ on a patch-wise uniform, regular mesh $\mathcal{T}_h$ of quadrilaterals in Ω admits a quantized tensor-train compression of accuracy $\varepsilon > 0$ in $H^1(\Omega)$ with tensor ranks bounded by $O(|\log \varepsilon|^5)$.

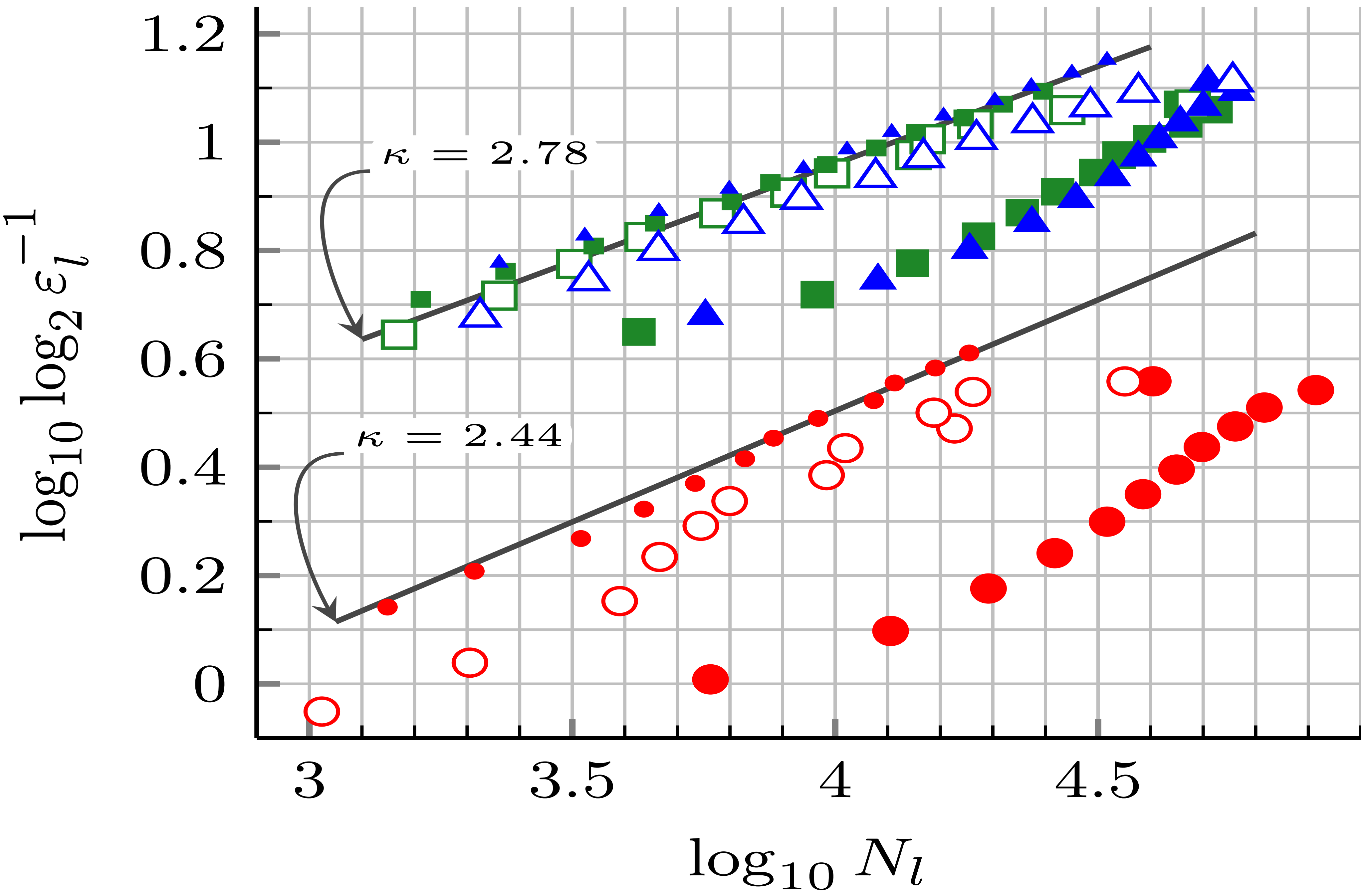
Remarks:

- this is a consistency result,
- proof by *nodal, bilinear interpolation approximation* of $u \in B_{-1-\boldsymbol{b}}(\Omega) \subset C^0(\overline{\Omega})$, error of tensor compressed interpolant by bilinear *reinterpolation of hp approximation of u*: exponential convergence.
- uses optimality (in ℓ^2), of quantization and HoSVD compression of the vector of function values,
- Computationally observed for QTT-FEM with quantized linear tensor system solver (AmEN iteration, TT Toolbox by I. Oseledets on GitHub).

Exponential Convergence of QTT h-FEM

Numerical Experiments (V. Kazeev):
Setting: Ω polygon in $\mathbb{R}^2$, Manufactured exact solution $u = r^\alpha \Phi(\alpha\theta)$.
Plot shows number N_l of quantized "tensor degrees of freedom" in continuous, piecewise bilinear FE approximation versus compression threshold ε_l, for scalar, linear elliptic $2d$ model problems.
Captions:

- circles: $\alpha = 1/4$, slit domain $\Omega = (0,1) \times (-1,1)$,
- squares: $\alpha = 2/3$, L-shaped domain,
- triangles: $\alpha = 3/4$, slit domain $\Omega = (0,1) \times (-1,1)$,
- large solid symbols: QTT-FE solutions
- small solid symbols: QTT interpolation approximations of u
- large empty symbols: recompressed QTT-FE solutions



1.2
1
0.8
0.6
0.4
0.2
0
3
3.5
4
4.5
$\log_{10} \log_2 \varepsilon_l^{-1}$
$\log_{10} N_l$
$\kappa = 2.78$
$\kappa = 2.44$

Conclusions

- Self contained proof of exponential convergence of hp Approximations on geometric meshes of hexahedra for solutions u of linear, 2nd order, elliptic problems subject to
- analytic regularity of u, and in axiparallel polyhedra.
- Analogous results expected in curved domains via *analytic patch maps*.
- σ geom. mesh and anisotropic polynomial degrees admissible
- hp Patch projectors for $\boldsymbol{c}$, $\boldsymbol{e}$ and $\boldsymbol{ce}$ patches, on H^1 with polynomial w.r. to p stability
- Exponential convergence of RB/MOR.
- Exponential convergence of QTT formatted, low-order FEM.
- hp-FEM a "background" method whose *approximation theory* initiated by Babuška and Guo serves as building block for "modern", dictionary based Galerkin projections.
- The groundbreaking results of Babuška and B.Q. Guo in the 80ies both in analysis (analytic regularity of elliptic PDEs) and FE approximation (exponential convergence of hp-FEM) continue to stimulate and drive current research in numerical analysis of PDEs in engineering and the sciences, worldwide.

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Happy 90th Anniversary, Ivo!